Elliptic Curves: The State of the Art

What Number Theorists Want to Know

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Outline:

- The Conjecture of Birch and Swinnerton-Dyer
- Ranks
- Integral Points
- Generalizations to Abelian Varieties
- Conclusions, Summary, References

Elliptic Curves over the Rationals

Mordell (1922): "Mathematicians have been familiar with very few questions for so long a period with so little accomplished in the way of general results, as that of finding the rational [points on elliptic curves]."

We still do not know an algorithm that is guaranteed to find the rational points on elliptic curves.

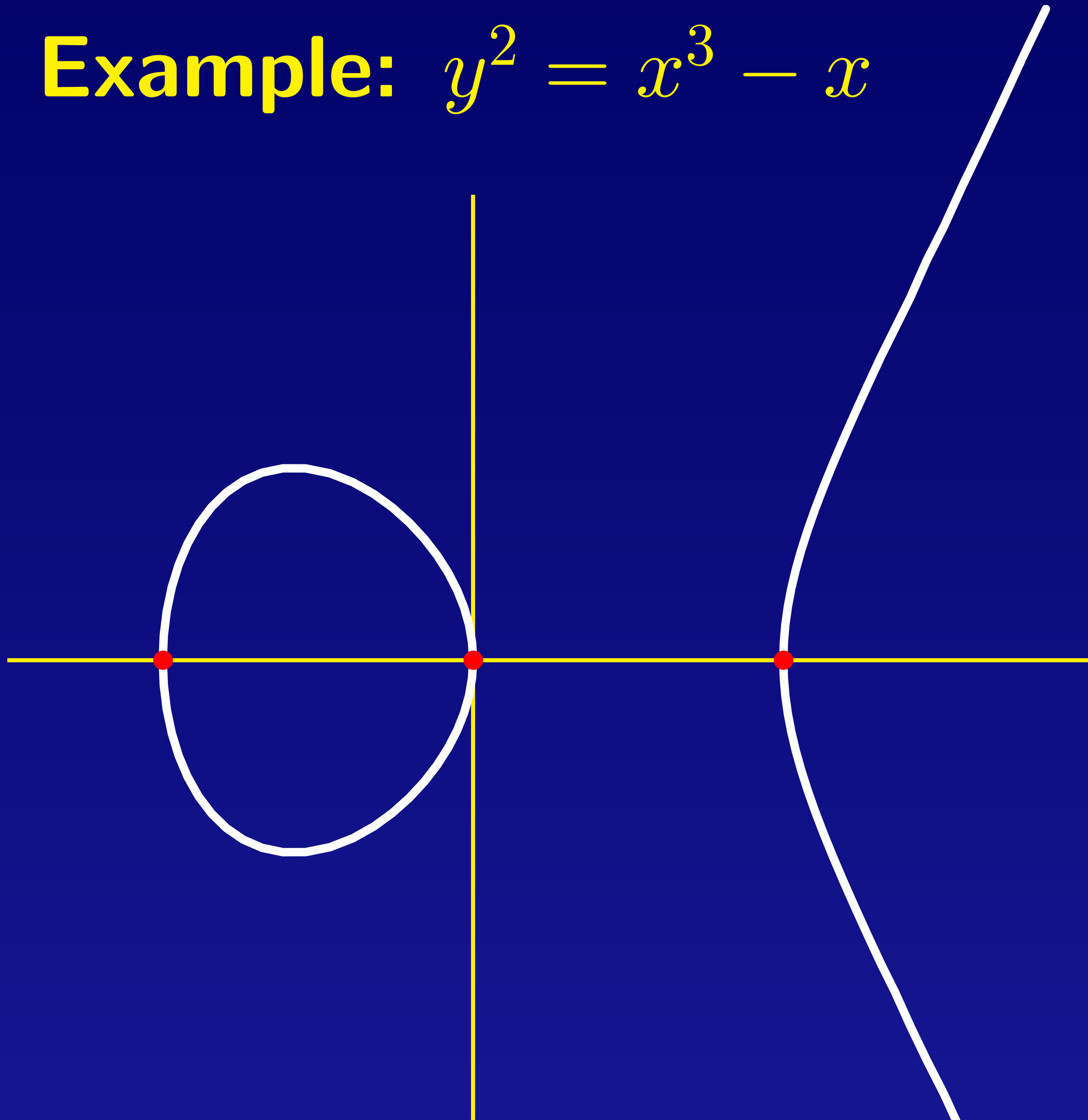
Elliptic Curves over the Rationals

$$E(\mathbb{Q}) \cong \text{ finite group } \times \mathbb{Z}^r$$
$$\cong E(\mathbb{Q})_{\text{tors}} \times \mathbb{Z}^r$$

$r =$ rank

Poincaré introduced ranks in 1901, and said they're clearly an interesting invariant to study.

Example: $y^2 = x^3 - x$



$$E(\mathbb{Q}) = \{(0,0), (1,0), (-1,0), O\} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

Hasse-Weil L-function

Take a minimal Weierstrass equation for E (coefficients in $\mathbb{Z}$, $|\Delta(E)|$ minimal).

Let $a_p = p + 1 - \#E(\mathbb{F}_p)$.

Let $L(E,s) = \displaystyle\prod_{p \nmid \Delta(E)} \frac{1}{1 - \frac{a_p}{p^s} + \frac{p}{p^{2s}}} \prod_{p \mid \Delta(E)} \frac{1}{1 - \frac{a_p}{p^s}}$.

The product converges absolutely for $\mathrm{Re}(s) > \frac{3}{2}$.

Hasse-Weil L-function

Theorem (Wiles, Taylor, Conrad, Diamond, Breuil). $L(E,s)$ *has an analytic continuation to the complex plane, and a functional equation relating* $L(E,s)$ *and* $L(E,2-s)$.

Hasse-Weil L-function

$$L(E,s) = b_r(s-1)^r + b_{r+1}(s-1)^{r+1} + \dots$$

$$b_i \in \mathbb{R},\ b_r \neq 0,\ r \in \mathbb{Z}^{\geq 0}.$$

This r is called the analytic rank of E (over $\mathbb{Q}$).

BSD, Part I

Conjecture. *rank = analytic rank*

Parity Conjecture

Conjecture BSD Part I $\Longrightarrow$

Parity Conjecture. *The rank and the analytic rank have the same parity.*

Congruent Number Problem

Conjecture BSD Part I implies an algorithm for solving the classical:

Congruent Number Problem. *Which positive integers are the areas of right triangles with rational sides?*

What's known about BSD Part I?

Theorem (Kolyvagin, Gross & Zagier,. . .).

(1) *If the analytic rank is* 0*, then the rank is* 0*.*

(2) *If the analytic rank is* 1*, then the rank is* 1*.*

BSD, Part II

Conjecture.

$$b_r = \frac{\Omega R \# Ш \prod_{p \mid \Delta(E)} c_p}{\#(E(\mathbb{Q})_{\mathrm{tors}})^2}$$

Recall:

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\mathrm{tors}} \times \mathbb{Z}^r$$

$$L(E,s) = b_r(s-1)^r + b_{r+1}(s-1)^{r+1} + \ldots$$

The Period

$$\Omega = \int_{E(\mathbb{R})} \frac{dx}{|2y + a_1 x + a_3|} \in \mathbb{R}$$

where

$$E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

$$b_r = \frac{\Omega R \# Ш \prod\limits_{p|\Delta(E)} c_p}{\#(E(\mathbb{Q})_{\mathrm{tors}})^2}$$

The Fudge Factors

c_p are small positive integers that measure the bad reduction of E at p.

$$(p \mid \Delta(E) \iff E \text{ has bad reduction at } p)$$

$$b_r = \frac{\Omega R \# Ш \prod_{p \mid \Delta(E)} c_p}{\#(E(\mathbb{Q})_{\mathrm{tors}})^2}$$

The Regulator

R measures the complexity of a minimal set of generators for $E(\mathbb{Q})$.

$$b_r = \frac{\Omega R \# \text{Ш} \displaystyle\prod_{p|\Delta(E)} c_p}{\#(E(\mathbb{Q})_{\text{tors}})^2}$$

The Tate-Shafarevich Group Ш

measures the failure of the Hasse Principle.

Hasse Principle (Local-to-Global Principle).
$\exists$ *points locally (over* $\mathbb{R}$ *and over all* $\mathbb{Q}_p$*)* $\Longrightarrow$
$\exists$ *points globally (over* $\mathbb{Q}$*).*

$$b_r = \frac{\Omega R \# \text{Ш} \displaystyle\prod_{p \mid \Delta(E)} c_p}{\#(E(\mathbb{Q})_{\mathrm{tors}})^2}$$

The Tate-Shafarevich Group Ш

Conjecture. Ш *is finite.*

Theorem (Cassels). *If* Ш *is finite, then* $\#$Ш *is a square.*

The Tate-Shafarevich Group Ш

Theorem (Rubin). *If the analytic rank is* 0*, and* E *has CM, then:*

(a) Ш *is finite,*

(b) *BSD Part II is true up to powers of* 2 *and* 3*.*

Theorem (Kolyvagin,. . .). *If the analytic rank is* 0 *or* 1*, then* Ш *is finite.*

BSD Part II: Example

$y^2 = x^3 - x \qquad\qquad \Delta(E) = 2^6$

$b_0 = L(E,1) \approx 0.655514\ldots \neq 0$
so analytic rank is 0.

Fermat: $E(\mathbb{Q}) = \{(0,0),(1,0),(-1,0),O\}$

BSD Part II: Example

$y^2 = x^3 - x \qquad\qquad \Delta(E) = 2^6$

$b_0 = L(E,1) \approx 0.655514\ldots \neq 0$
so analytic rank is 0.

Fermat: $E(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \cong E(\mathbb{Q})_{\mathrm{tors}}$ so
rank $= 0$ and $\#E(\mathbb{Q})_{\mathrm{tors}} = 4$.

$R = 1, \quad \Omega \approx 5.2441\ldots, \quad c_2 = 2$

Rubin proved $Ш = 0$.

B&SD proved $b_0 = \frac{\Omega}{8}$.

BSD Parts I and II are true.

BSD Part II: Example

$y^2 = x^3 - 25x \qquad \Delta(E) = 2^6 \cdot 5^6$

$b_0 = L(E,1) = 0, \quad b_1 = L'(E,1) \approx 2.227\ldots \neq 0,$
so analytic rank is 1.

A descent shows $E(\mathbb{Q})$ is generated by $P = (-4,6)$ and points of order 2 so rank is 1, $\#E(\mathbb{Q})_{\mathrm{tors}} = 4$.

$\Omega \approx 2.3452\ldots, \quad R \approx 1.89948\ldots, \quad c_2 = 2, \quad c_5 = 4$

Kolyvagin proved $Ш = 0$.

Gross & Zagier proved $b_1 = \frac{\Omega R}{2}$.

BSD Parts I and II are true.

Ranks

Recall:

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{\mathrm{tors}} \times \mathbb{Z}^r$$

$r =$ rank

There is no known algorithm guaranteed to determine the rank.

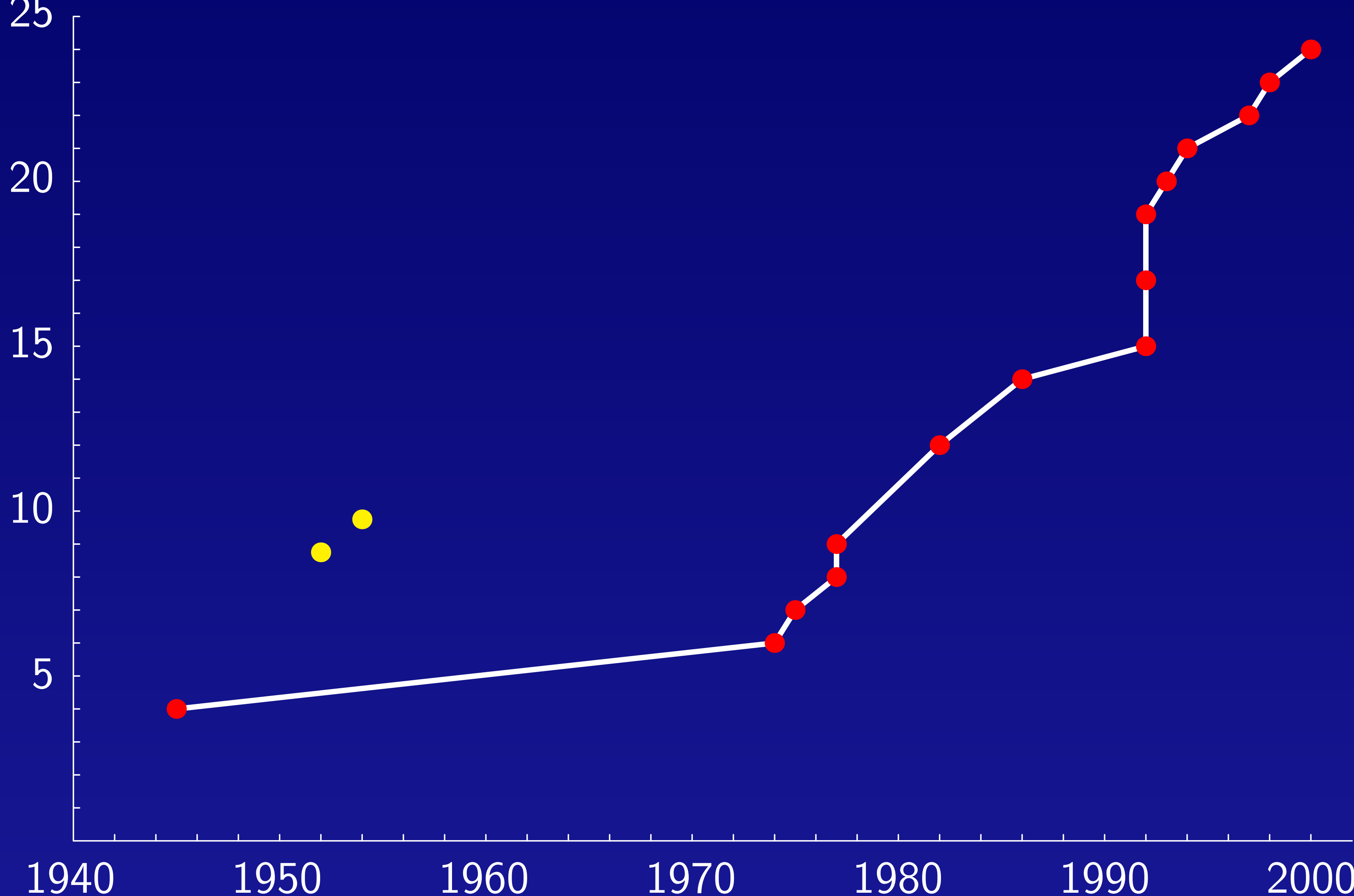
It is not known which integers can occur as ranks.

It is not known if ranks are unbounded.

Rank Records

Rank $\geq$	Year	Discoverers
3	1945	Billing
4	1945	Wiman
6	1974	Penney & Pomerance
7	1975	Penney & Pomerance
8	1977	Grunewald & Zimmert
9	1977	Brumer & Kramer
12	1982	Mestre
14	1986	Mestre
15	1992	Mestre
17	1992	Nagao
19	1992	Fermigier
20	1993	Nagao
21	1994	Nagao-Kouya
22	1997	Fermigier
23	1998	Martin-McMillen
24	2000	Martin-McMillen

Rank Records



Rank Records

Martin and McMillen's curve with rank at least 24:

$$y^2 + xy + y = x^3 - 120039822036992245303534619191166796374x+$$
$$504224992484910670010801799168082726759443756222911415116.$$

Quadratic Twists

$E : y^2 = f(x)$

$E_d : dy^2 = f(x)$ quadratic twist by d

Honda's Conjecture. *Ranks are bounded in families of quadratic twists.*

Ranks in the family $dy^2 = x^3 - x$

rank	d	d factored	person	year
0	1	1	Fermat	~1640
1	5	5	Billing	1937
2	34	$2 \cdot 17$	Wiman	1945
3	1254	$2 \cdot 3 \cdot 11 \cdot 19$	Wiman	1945
4	29274	$2 \cdot 3 \cdot 7 \cdot 17 \cdot 41$	Wiman	1945
5	205015206	$2 \cdot 3 \cdot 11 \cdot 17 \cdot 19 \cdot 59 \cdot 163$	Rogers	2000
6	61471349610	$2 \cdot 3 \cdot 5 \cdot 11 \cdot 19 \cdot 41 \cdot 43 \cdot 67 \cdot 83$	Rogers	2000

Wiman said that if d with rank greater than 4 exist in this family, they would be almost insurmountably difficult to find.

Rogers used a clever computer search, using ideas of Silverberg and Rubin.

A Density Conjecture

Fix E.

Parity Conjecture $\Longrightarrow$

A Density Conjecture.

(i) $\frac{1}{2}$ *the* E_d*'s have even rank;*

(ii) $\frac{1}{2}$ *the* E_d*'s have odd rank;*

(iii) *average rank of the* E_d*'s is* $\geq \frac{1}{2}$*.*

Goldfeld Conjecture

Conjecture (Goldfeld).
Average rank of the E_d's is $\frac{1}{2}$.

Goldfeld + Parity Conjectures $\Longrightarrow$

Density Conjecture.

(i) $\frac{1}{2}$ *the* E_d*'s have rank* 0*;*

(ii) $\frac{1}{2}$ *the* E_d*'s have rank* 1*;*

(iii) *density* 0 *have rank* ≥ 2*.*

Density

$$N_*(X) := \#\{\text{squarefree } d \in \mathbb{Z} : |d| \le X,$$
$$\text{rank of } E_d \text{ is } *\}$$

$$N_{\geq 0}(X) \sim \frac{2}{\zeta(2)} X = \frac{12}{\pi^2} X$$

Parity Conjecture $\Longrightarrow$

$$N_{\text{odd}}(X) \sim N_{\text{even}}(X) \sim \tfrac{1}{2} N_{\geq 0}(X)$$

Goldfeld + Parity Conjectures $\Longrightarrow$

$$N_0(X) \sim N_1(X) \sim \tfrac{1}{2} N_{\geq 0}(X), \quad N_{\geq 2}(X) = o(X)$$

Density

Theorem (Gouvêa & Mazur, Stewart & Top, Rubin & Silverberg).

(i) $N_{\geq 1}(X) \gg X^{1/2}$.

(ii) *For certain* E, $N_{\geq 2}(X) \gg X^{1/3}$.

(iii) *For a smaller class of* E, $N_{\geq 3}(X) \gg X^{1/6}$.

Density

Theorem (Gouvêa & Mazur, Stewart & Top, Rubin & Silverberg).

Parity Conjecture $\Longrightarrow$

(iv) $N_{\geq 2}(X) \gg X^{1/2}$.

(v) *For certain* E, $N_{\geq 3}(X) \gg X^{1/3}$.

(vi) *For a smaller class of* E, $N_{\geq 4}(X) \gg X^{1/6}$.

Some Connections with ECC

- Silverman's Xedni Calculus attack on ECC is based on work of Mestre on finding ECs with high rank.

  (See Jacobson-Koblitz-Silverman-Stein-Teske for an analysis of the attack.)

- Huang et al. (ANTS IV) have shown a relationship between the (conjectured) unboundedness of ranks of ECs and attacks on ECC.

Integral Points

Conjecture (Lang). *There is an absolute constant* C *such that if* E *is given by a minimal (affine) Weierstrass equation, then the number of integral points is at most* $C^{1+\text{rank of } E}$.

Theorem (Silverman). *True if* E *has everywhere potentially good reduction (i.e.,* $j(E) \in \mathbb{Z}$*).*

Abelian Varieties

A an abelian variety over a number field F

$$A(F) \cong A(F)_{\mathrm{tors}} \times \mathbb{Z}^r$$

There are Birch and Swinnerton-Dyer Conjectures for abelian varieties.

Not much is known about ranks of abelian varieties.

Torsion Conjecture for Abelian Varieties

Torsion Conjecture. *$\#A(F)_{\mathrm{tors}}$ is bounded above by a constant depending only on the degree of F and the dimension d of A.*

Proved for $d = 1$ and $F = \mathbb{Q}$ by Mazur.

Proved for $d = 1$ by Merel.

Open for abelian varieties of dimension > 1.

Abelian Varieties and Modularity

Modularity Conjecture. *Every elliptic curve over* $\mathbb{Q}$ *is modular.*

(was proved by Wiles, Taylor, Conrad, Diamond, and Breuil)

There's a Modularity Conjecture for (certain) abelian varieties.

Open Questions: BSD

BSD I: rank = analytic rank

- Parity Conjecture: rank and analytic rank have same parity
- Congruent Number Problem

BSD II

- Finiteness of Ш

Open Questions: Ranks

- Which integers can occur as ranks?
- Unboundedness of ranks

In families of quadratic twists:

- Unboundedness of ranks
- Goldfeld's Conjecture: average rank $= \frac{1}{2}$
- Density Conjecture: $\frac{1}{2}$ the ranks are 0, $\frac{1}{2}$ are 1
- Find behavior of N_r's ($\#$ of d with rank r)

Open Questions: Integral Points

- Lang's Conjecture: bound the number of integral points in terms of the rank

Open Questions: Abelian Varieties

- BSD
- Ranks
- Torsion
- Modularity

Reference

A. Silverberg, *Open questions in arithmetic algebraic geometry*, in Arithmetic Algebraic Geometry (Park City, UT, 1999), IAS/Park City Mathematics Series **9**, AMS, Providence, RI (2001).

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No Microsoft products were used in this presentation.