Computing discrete logs in large characteristic 2 finite fields

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Plan

1. Introduction – Finite field discrete logs
2. Subexponential algorithms for $\mathbb{F}_{2^n}$
3. Different sieve techniques
4. Linear algebra
5. Using and improving the Block Wiedemann algorithm
6. Runtime figures: $\mathbb{F}_{2^{607}}$
7. Conclusion

1. Introduction

Discrete logarithms (DL) appear in several cryptographical settings:

- Diffie-Hellman key exchange.
- El Gamal cryptosystem, El Gamal & Schnorr signature scheme.
- Most ID-based cryptosystems are based on DL instances.

Different groups have been proposed for implementation:

- Finite fields.
- (hyper-)Elliptic curves . . .
- XTR.

Finite field DL : What for ?

- Attack cryptosystems directly based on finite field DL.
- With the Tate pairing, for E elliptic curve over $\mathbb{F}_q$, DL on E reduces to DL on $\mathbb{F}_{q^k}$, with k s.t. $\#E(\mathbb{F}_q)|q^k - 1$.
  (MOV reduction on supersingular elliptic curves).

Accounts on the (in)feasibility of DL computations in given finite fields are of high interest to many cryptosystems, including recent ID-based schemes.

Finite field DL (2)

Subexponential algorithms for this task:

- $\mathbb{F}_p$: Number field sieve (Gordon, Schirokauer)
- $\mathbb{F}_{2^n}$: Function field sieve (Adleman), Coppersmith

Common heuristic complexity:

$$L[1/3] = O(\exp((c + o(1))(\log \#K)^{1/3}(\log\log \#K)^{2/3})).$$

Which field sizes can be reached today ?

We focus on characteristic 2 fields,
trying to improve on Coppersmith's algorithm.

Record presented: $\mathbb{F}_{2^{607}}$
Another recent computation: $\mathbb{F}_{2^{521}}$ by Joux and Lercier, using FFS (Function field sieve).

2. Coppersmith's algorithm for $\mathbb{F}_{2^n}$

$\mathbb{F}_{2^n} = \mathbb{F}_2[X]/(f(X))$, where $f(X) = X^n + f_1(X)$ irreducible, $\deg f_1$ small.

Index-calculus type algorithm:

- Choose a factor base: $\mathcal{B} = \{\pi_i \in \mathbb{F}_2[X],\ \pi_i \text{ irreducible},\ \deg \pi_i \leq b\}$.
- Gather sufficiently many ($\frac{2^{b+1}}{b}$) relations between the π_i's.
- Obtain the $\log \pi_i$ as the solution of a linear system.
- $\mathcal{B}$ is chosen large enough so that it is then easy to express the log of anything as a linear combination of the $\log \pi_i$.

Building relations in Coppersmith's algorithm

Consider the pairs (A, B) of polynomials of degree $\leq d$ (here, $d = 28$). Define:

$$C = AX^h + B,$$

$$D = A^k X^{hk-n} f_1 + B^k, \text{ where } n = 607, k = 4, h = 152.$$

One easily verifies that $C^k \equiv D \bmod f(X)$. $\deg C \leq 180, \deg D \leq 112$. If both C and D are b-smooth (all their factors have degree $\leq b$):

$$\prod_i \pi_i^{ke_i} \equiv \prod_i \pi_i^{f_i} \mod f(X),$$

$$\sum_i (f_i - ke_i) \log \pi_i \equiv 0 \mod 2^n - 1.$$

Choosing parameters b, d, is far from easy. Here we took $b = 23$, hence $\#\mathcal{B} = 766,150$.

Parallel with the Function Field Sieve

Coppersmith's algorithm is a special case of the Function Field Sieve.

Let H be the curve over $\mathbb{F}_2$ defined by $H(X,Y) = Y^k + X^{hk-n} f_1(X)$.

Let F be the field of rational functions on H: $F = \mathbb{F}_2(X,Y)/H(X,Y)$.

$$\Pi : \begin{cases} F & \rightarrow & \mathbb{F}_{2^n} = \mathbb{F}_2[X]/f(X), \\ Y & \mapsto & X^h. \end{cases}$$

Consider a function on H: $\phi = A(X)Y + B(X) \in F$. We are interested in:

- Smoothness of $\Pi(\phi) = A(X)X^h + B(X)$ as a polynomial in X.
- Smoothness of ϕ in F (as a sum of principal divisors). Its norm is:

$$A(X)^k H(X, \frac{B(X)}{A(X)}) = A(X)^k X^{hk-n} f_1(X) + B(X)^k.$$

In this context, the choice of $H(X,Y)$ can be refined.

In short: for a good (A, B) pair, C and D are smooth $\Rightarrow$ one relation.

We concentrate on:

How to efficiently obtain relations (find good (A, B) pairs)?

How to solve the resulting linear system?

3. Building relations: different sieving techniques

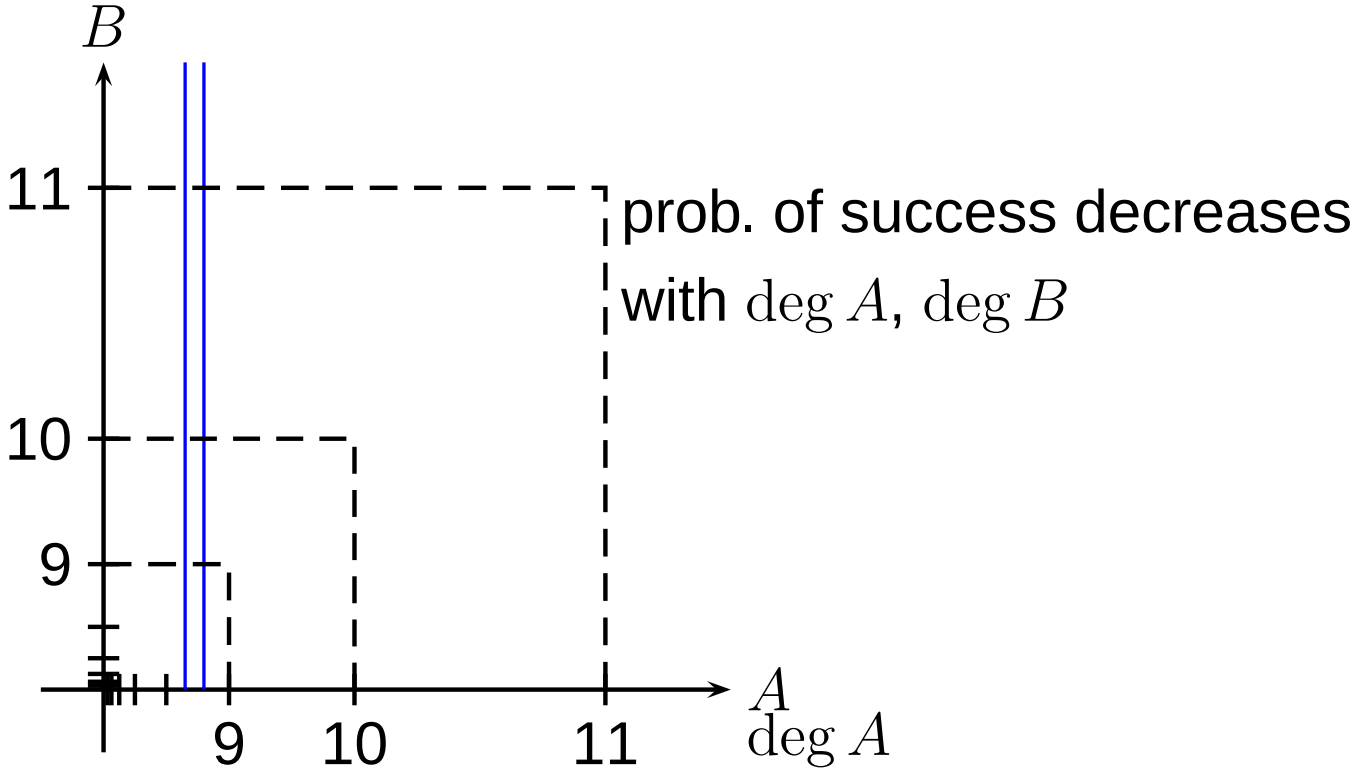
It is good to sieve in order to find appropriate (A, B) pairs: restrict our scope to the pairs where C has a good chance of being smooth.

First method: **fix A**, and try to spot B's such that (A, B) yields a smooth C. For each irreducible g in the factor base, consider:

$$\begin{aligned}\mathcal{S}(g) &= \{B \text{ s.t. } C = AX^h + B \equiv 0 \bmod g\},\\ &= \text{affine subspace of } < 1, X, \ldots, X^d >,\\ &= (AX^h \bmod g) + < g, Xg, \ldots, X^{d-\deg g} g > .\end{aligned}$$

Have a table of integers associated to B's, and add $\deg g$ to all entries meeting $\mathcal{S}(g)$. At the end, entries with highest values are likely to yield good pairs.

The sieving range



Typically, this sieving task is distributed across a huge number of machines, of different computational power. We want to bring more flexibility to this scheme.

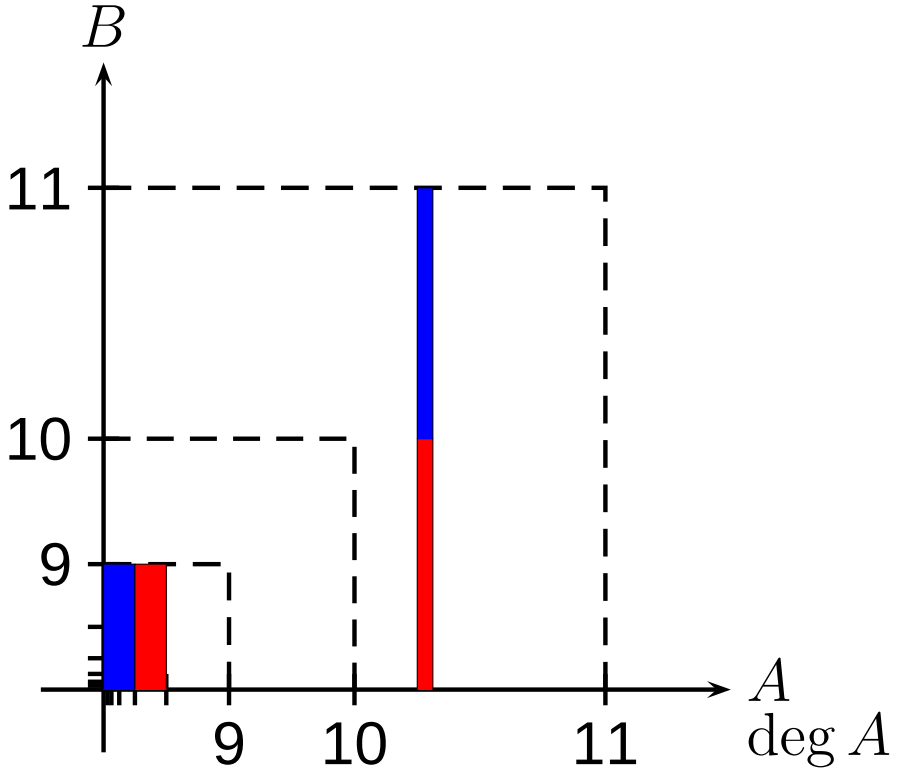
An improvement to the standard sieve: sieve grouping

We no longer fix all the bits of A, but allow some (ϵ) of them to vary.
Call A_f the **fixed part** of A.
Write $A = A_f + A_v$, $X^\epsilon | A_f$, $\deg A_v < \epsilon$. For each g, we consider:

$$\begin{aligned}
\mathcal{S}(g) &= \{(A_v, B) \text{ s.t. } C = (A_f + A_v)X^h + B \equiv 0 \bmod g\}, \\
&= \text{affine subspace of } < 1, X, \ldots, X^d, X^h, \ldots, X^{h+\epsilon-1} >, \\
&= (A_f X^h \bmod g) + \\
&\quad < g, Xg, \ldots, X^{d-\deg g} g, X^h + (X^h \bmod g), \ldots > .
\end{aligned}$$

$\mathcal{S}(g)$ is almost as easy to compute as previously, but provides initialization data for 2^ϵ as much sieve space.

The sieving range with sieve grouping



- Start with the most interesting part of the sieve range.
  No loss of time on the less interesting part.
  $\Rightarrow$ more flexibility, better usage of the sieve range.
- Large RAM $\Rightarrow$ reduce the initialization cost to almost nothing.
  It pays off to set up initialization data for $\epsilon > 0$, and then do sub-sieves.

Many other sieve improvements are possible

- Use of large primes (1, 2, more): "Quasi-relations" involving large cofactors can be combined to yield plain smooth relations. We used double large prime variation. Complications arise because "large primes" must be considered with their exponent.
- Avoid sieving for too small or too large g's, to cut down on the sieving and initialization time. We only sieved for $10 \leq \deg g \leq 22$.
- Do special-q (lattice) sieving: use linear algebra over $\mathbb{F}_2$ to compute the initialization of the sieve in that case (not extensively used here).
- Sieve over D if useful (it turned out not to be).
  Initialization harder $\Rightarrow$ Sieve grouping pays off even more then.

4. The resulting linear system

We have a collection of relations of the form:

$$\sum_i (f_i - ke_i) \log \pi_i \equiv 0 \mod q = 2^n - 1.$$

$\Rightarrow$ singular $N \times N$ matrix M. Look for $w \in (\mathbb{Z}/q\mathbb{Z})^N,\ Mw = 0$.

This matrix is large and sparse: Few ($\gamma \ll N$) non-zero coefficients per row.

- Gaussian elimination: $O(N^3)$.
- Dedicated algorithms: $O(\gamma N^2)$ $\Rightarrow$ must be employed.

Two important stumbling blocks:

- The base ring is very large: $\mathbb{Z}/(2^n - 1)\mathbb{Z}$
- Not all the coefficients are ± 1 (k is 4).

Dealing with the sparse matrix

First pass: structured gaussian elimination. This procedure “crunches” the matrix, trying to keep it as sparse as possible.

- An estimate of the cost of handling the resulting (still sparse) matrix afterwards is kept, in order to have this estimate minimized.
- While doing this structured gaussian elimination, some rows are removed from the system. Heuristics on how to do this were carefully analyzed, and the influence of this on the resulting matrix is very sensible.
- Not all coefficients are ± 1, so pivoting must avoid excessive coefficient growth.

5. Using and improving the block Wiedemann algorithm

After the structured gaussian elimination, the matrix is still sparse.

Dedicated algorithms available to compute an element of the nullspace:

- (block) Lanczos.
- (block) Wiedemann.

We used the block Wiedemann (BW) algorithm.

BW accomplishes a step towards partial distribution of the linear algebra. However, there is still a sequential step (improved here).

The computation of an element of the nullspace is done modulo each of the large prime factors of $2^n - 1$, and the result is recovered by Chinese remaindering.

Principle of the Wiedemann algorithm

- Given a singular matrix M of size $N \times N$ over K, choose random vectors $x, y \in K^N$, and compute:

$$A(X) = \sum_{i=0}^{2N-1} a_i X^i = \sum_{i=0}^{2N-1} (x^T M^{i+1} y) X^i \in K[X].$$

- Try to compute a rational fraction expansion of $A(X)$, via HGCD or Berlekamp-Massey algorithm.

$$A(X)F(X) = G(X) + O(X^{2N}),\ \deg F \leq N,\ \deg G < N.$$

$\hat{F}(X)$ (the reciprocal of F) is then a divisor of the minimum polynomial of M, and $\hat{F}(M)y$ is in the nullspace of M.

Complexity: $2N + N = 3N$ matrix-times-vector products, $+O(N^2)$ or less for the Berlekamp-Massey algorithm.

Principle of the block Wiedemann algorithm

- Replace vectors x, y by blocks of vectors $x \in K^{N \times m}$, $y \in K^{N \times n}$, with m and n two chosen integers.
- $A(X)$ becomes a matrix of polynomials:

$$A(X) = \sum_{i=0}^{L} a_i X^i = \sum_{i=0}^{L} (x^T M^{i+1} y) X^i \in K^{m \times n}[X], L = \frac{N}{m} + \frac{N}{n} + \epsilon.$$

- We still look for $F(X) \in K[X]^n$ such that:

$$A(X)F(X) = G(X) + O(X^L),\ \deg F \leq \frac{N}{n},\ \deg G < \frac{N}{n}.$$

- As before, once $F(X)$ is obtained, an element of the nullspace can be found.

What are the differences with the “plain” Wiedemann algorithm?

Advantages of BW over plain Wiedemann (1)

Computation of $A(X) = \sum_{i=0}^{L} (x^T M^{i+1} y) X^i$ is easier.

Column k of $A(X)$ depends uniquely on column k of y.

If computers $C_0, \ldots, C_{n-1}$ each hold a copy of M and one column y_k of y, computer C_k iterates:

- $y_k \leftarrow M y_k$.
- Output $x^T y_k$.

These computations can take place on different computers, with zero communication.

Advantages of BW over plain Wiedemann (2)

Computation of $F(X)$ s.t. $A(X)F(X) = G(X) + O(X^L)$ is harder.

- Matrix context:
  - $A(X)$: an $m \times n$ matrix of polynomials: $A(X) \in K[X]^{m \times n}$
  - $F(X)$: an column vector of n polynomials: $F(X) \in K[X]^{n \times 1}$
  - $G(X)$: an column vector of m polynomials: $G(X) \in K[X]^{m \times 1}$

  $\Rightarrow$ We need a matrix version of the Berlekamp-Massey algorithm.
- Sequential nature apparently unavoidable.
- Base field is $\mathbb{F}_p$, not $\mathbb{F}_2 \Rightarrow$ much harder.

Coppersmith suggested one matrix generalization of Berlekamp-Massey, in time $O(N^2)$.

We provide an asymptotically fast, FFT-based extension, in $O(N \log^2 N)$. It turned out to be a big improvement.

Subquadratic matrix Berlekamp-Massey (1)

We consider "candidate" solutions, for increasing values of t:

$$A(X)F(X) = G(X) + X^t E(X),\ \deg F \leq \delta,\ \deg G < \delta.$$

- Our goal: $t \geq L$ and $\delta \leq \frac{N}{n}$.
- For each t, a tuple of $m+n$ "candidates": $(f_j, g_j, e_j, \delta_j)_{j=0\ldots m+n-1}$ gathered to form matrices: $f(X) \in K[X]^{n\times(m+n)}$, $g(X) \in K[X]^{m\times(m+n)}$, $e(X) \in K[X]^{m\times(m+n)}$
- Setup: technical but easy, $t_0 \in O(1)$, and $\forall j, \delta_j = t_0$.

Going from t to $t+1$:

$e(0)$ known $\rightsquigarrow$ $P^{(t)}$ of degree 1 s.t:

$$f^{(t+1)} = f^{(t)} P^{(t)}$$
$$g^{(t+1)} = g^{(t)} P^{(t)}$$
$$e^{(t+1)} = e^{(t)} P^{(t)} \frac{1}{X}$$

We can work with $e(X)$ only

Subquadratic matrix Berlekamp-Massey (2)

Fact: Partial knowledge of $e(X)$ (e.g. $e(X) \bmod X^b$) is enough to:

- compute $P^{(t)}(X)$,
- deduce $e^{(t+1)}(X) \bmod X^{b-1}$, since $e^{(t+1)} = e^{(t)} P^{(t)} \frac{1}{X}$.

$\Rightarrow$ Divide-and-conquer approach. To compute $f^{(L)}(X)$:

- Compute $e^{(0)}(X) \bmod X^L$, go into recursive procedure:
  - Using $e^{(0)}(X) \bmod X^{\frac{L}{2}}$, compute recursively $P^{(0)} \dots P^{(\frac{L}{2}-1)}$,
  - Deduce $e^{(\frac{L}{2})}(X) \bmod X^{\frac{L}{2}}$, compute recursively $P^{(\frac{L}{2})} \dots P^{(L-1)}$.
  - Compute $P^{(0)} \dots P^{(L-1)}$
- Then $f^{(L)}(X) = f^{(0)}(X) P^{(0)} \dots P^{(L-1)}$.

Multiplication of large polynomial matrices done efficiently with FFT.

6. The record computation over $\mathbb{F}_{2^{607}}$

Factor base size: 766,150 polynomials ($b = 23$).

Sieving code:

- Idle time was used on a large ($\sim$ 100) number of linux PCs and alphas (Typical machine: Pentium II 450 MHz).
- Code was run from late-2000 to mid-2001.
  19,000 MIPS years equivalent (?)
- Double large prime variation used
- More than 10GB of partial relations gathered.
- At the end, 1,033,593 relations obtained, with more that 80% from large primes.

The linear algebra step for $\mathbb{F}_{2^{607}}$

Linear system obtained: 1,033,593 equations, 766,150 unknowns, $\sim$ 70,000,000 non-zero coefficients.

After the structured gaussian elimination (a few hours CPU):

- Matrix of size 480,000$\times$480,000,
- 107 coeffs/row on average.
- RAM storage for the matrix: 400MB.

Block Wiedemann algorithm done using several SMP machines:

- We used several unconnected machines
- We also used the SMP capabilities of each machine (POSIX threads, etc. . .)

. . . /. . .

The linear algebra step for $\mathbb{F}_{2^{607}}$ (continued)

- Non-sequential steps: 18 days on 6 4-cpu 667MHz EV67's
- Sequential step: 4 days on one 667MHz EV67 cpu.
  - Nearly 2 months of computation time saved here with our new recursive approach.
  - Enormous polynomial matrix multiplications with FFT required very large swap memory (20GB).
- Interest of having used the block algorithm:
  total time of 3 weeks instead of 4 months !

The computation of an individual log takes 1 to 2 hours CPU on a 700MHz Pentium III.

7. Conclusion

- A battery of improvements have been used to improve the sieve part: sieve grouping, incomplete sieves (skipping low and high degree g's), (double) large primes, special-q sieving.
- Some aspects have not been used, or not extensively: multiple (> 2) large primes schemes, special-q sieving
- The linear algebra benefited from heuristics for the structured gaussian elimination, and an improvement of the block Wiedemann algorithm.
- This is a very long computation, involving a fair amount of both human time and computational power.
- Access to a private, closed network of computers is bound to reduce drastically the computation time.