An $L(1/3)$ algorithm for the discrete logarithm problem in low degree curves

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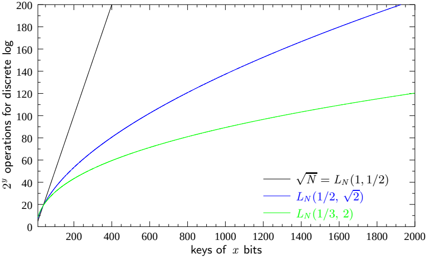
The discrete logarithm problem (DLP)

Given $b = a^x$ in some group, find x.
Given $b = x \cdot a$ in some group, find x.



The subexponential function L

$$L_N(\alpha, c) = e^{(c+o(1))(\log N)^\alpha (\log\log N)^{1-\alpha}},\ \alpha \in (0,1),\ c > 0$$

- $\alpha = 0$: $(\log N)^c$
- $\alpha = 1$: N^c

- $L_N(\alpha, c_1) \cdot L_N(\alpha, c_2) = L_N(\alpha, c_1 + c_2)$
- $(\log N)^k \in L_N(\alpha, 0)$

- $L_{q^g}(\alpha, c) \approx q^{cg^\alpha} \approx q^{g^\alpha}$ number of elements
- $\log_q L_{q^g}(\alpha, \cdot) \approx g^\alpha$ degree of elements

$L(1/3)$

1. Discrete logarithms in $L(1/2)$
   - $\mathbb{F}_{2^g}$
   - Algebraic curves
2. Finding relations in $L(1/3)$
   - $\mathbb{F}_{2^g}$
   - Algebraic curves
3. Computing discrete logarithms
   - Optimally unbalanced curves
   - More balanced curves

Subexponential algorithm for $\mathbb{F}_{2^g}$ — ingredients

Problem: Given $b = a^x \in \mathbb{F}_{2^g}^\times$, find x; $\mathbb{F}_{2^g} = \mathbb{F}_2[X]/(f(X))$

- elements

  polynomials over $\mathbb{F}_2$ of degree $< g$
- prime elements

  irreducible polynomials
- size function $\longrightarrow \mathbb{R}^+$, homomorphic

  $\deg$
- factor base of prime elements up to some smoothness bound B

$$\mathcal{F} = \{p_1, \ldots, p_n\}$$

- non-unique decomposition into prime elements

$$\begin{aligned} r(X) &= \prod p_i^{e_i} \\ r(X) + (X^4+1)f(X) &= \prod p_i^{f_i} \\ \text{relation} \quad \prod p_i^{e_i - f_i} &= 1 \end{aligned}$$

Subexponential algorithm for $\mathbb{F}_{2^g}$

- Take random $\alpha_j, \beta_j \in \mathbb{Z}$ and compute

  $a^{\alpha_j} b^{\beta_j} \bmod f$, a polynomial over $\mathbb{F}_2$ of degree $< g$
- Sometimes, the result is $\mathcal{F}$-smooth (or B-smooth)

$$\begin{aligned} a^{\alpha_j} b^{\beta_j} &= \prod_{i=1}^{n} p_i^{\alpha_{ij}} \\ \alpha_j + \beta_j x &= \sum \alpha_{ij} \log_a p_i \end{aligned}$$

- Linear algebra yields x (and the $\log_a p_i$)

Complexity

Depends on the choice of $\mathcal{F}$

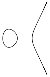
- $\mathcal{F}$ too small
  - small probability of smoothness
- $\mathcal{F}$ too large
  - too many relations needed
  - linear algebra infeasible
- good compromise:
  - $B = \log_2 L_{2^g}(1/2, \sqrt{2}/2) \approx g^{1/2}$
  - $|\mathcal{F}| = L_{2^g}(1/2, \sqrt{2}/2) \approx 2^{g^{1/2}}$
  - smoothness probability $1/L_{2^g}(1/2, \sqrt{2}/2)$
- total complexity

$$L_{2^g}(1/2, \sqrt{2})$$

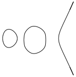
$L(1/3)$

"Simple" algebraic curves

- elliptic curves
  $Y^2 = X^3 + aX + b$, $g = 1$

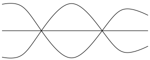


- hyperelliptic curves of genus g
  $Y^2 = f(X) = X^{2g+1} + \cdots$

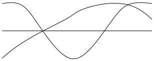


"Less simple" algebraic curves

- superelliptic curves
  $Y^n = f(X) = X^d + \cdots$, $g = \frac{(n-1)(d-1)}{2}$
  in particular: $Y^3 = X^4 + f_3 X^3 + f_2 X^2 + f_1 X + f_0$, $g = 3$



- $C_{n,d}$ curves
  $Y^n + h(X, Y) = X^d$, terms in h of small degree, $g = \frac{(n-1)(d-1)}{2}$
  in particular: $Y^3 + h(X)Y = X^4 + f(X)$, $\deg h \leq 2$, $\deg f \leq 3$



Divisors over $\overline{\mathbb{F}_q}$

- divisors = finite formal sums of points
  $D = \sum_{P \in C} m_P P$, $m_P \in \mathbb{N}$, almost all zero
- prime elements = points
- Mumford representation

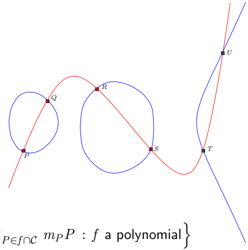
$$\begin{aligned} D &= (x_1, y_1) + \cdots + (x_g, y_g) \\ &= (u, Y - v), \\ &\quad u = (X - x_1) \cdots (X - x_g), \\ &\quad v \text{ of degree } g - 1 \text{ s.t. } v(x_i) = y_i \end{aligned}$$

- prime elements = irreducible u
- adding

$$(u_1, v_1) + (u_2, v_2) = (u_1 u_2, v_3)$$

  with v_3 the Lagrange interpolation polynomial (extended Euclidian algorithm)
- decomposition = factoring u

Jacobians



- $\mathrm{Prin} = \left\{ \sum_{P \in f \cap \mathcal{C}} m_P P \;:\; f \text{ a polynomial} \right\}$
- $J = \mathrm{Div} \,/\, \mathrm{Prin}$
- $(P + Q) + (R + S) = (-T) + (-U) = \overline{T} + \overline{U}$ in J (reduction)
- non-unique prime decomposition

Jacobians over $\mathbb{F}_q$

- prime divisor = orbit of points under the Galois group (Frobenius)
- degree = number of points in the divisor
- example
  - $(x, y) \in \mathbb{F}_{q^k} \times \mathbb{F}_{q^k}$
  - set of k points (x^{q^i}, y^{q^i})
  - prime divisor of degree k:
    $(u, \, Y - v)$, u minimal polynomial of x over $\mathbb{F}_q$
- adding
  $\sum (u_i, \, Y - v_i) = (u, \, Y - v)$
  - $u = \prod u_i$
  - v s.t. $v \equiv v_i \bmod u_i$

  + reduction
- prime decomposition = factoring u
  $(u, \, Y - v) = \sum (u_i, \, Y - v_i)$
  - $u = \prod u_i$
  - $v_i = v \bmod u_i$

Subexponential algorithm

- Riemann–Roch: $\deg u \leq g$
- Hasse-Weil:
  - number of points over $\mathbb{F}_{q^k} \approx q^k$
  - number of prime divisors of degree $k \approx q^k / k$
  - $\# J \approx q^g = q^{g^1} \approx L_{q^g}(1, \cdot)$

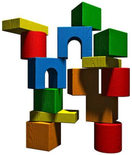
The algorithm of $\mathbb{F}_{2^g}$ applies...
... and has complexity $L_{q^g}(1/2, \sqrt{2})$

- at least for q fixed, $g \to \infty$
- more precisely for $g > \log q$

History

- Adleman–DeMarrais–Huang 1994
  - first subexponential algorithm for hyperelliptic curves
  - heuristic
- Müller–Stein–Thiel 1999
  - algorithm for infrastructure of real quadratic function fields
  - heuristic, since smoothness result missing
- E. 2002
  - first subexponential algorithm for hyperelliptic curves
    with proven complexity (smoothness result in E.–Stein 2002)
- E.–Gaudry 2002
  - unified framework for discrete logarithm algorithms in $L(1/2)$
    (finite fields, class groups of number fields, Jacobians)
- Couveignes 2001, Hess 2004
  - proven $L(1/2)$-algorithms for all major classes of curves
- exponential, but fast algorithms for smallish genus

Construction kit smoothness result



- Assume there are $\approx q^k/k$ bricks of size k.
- Consider random towers of height $\log_q L_{q^g}(\alpha, \cdot)$.
- Interest yourself in those constructed from small bricks of size up to $\log_q L_{q^g}(\beta, \cdot)$.
- Then their proportion is

$$1/L_{q^g}(\alpha - \beta, \cdot)$$

- Application: $\alpha = 1$, $\beta = 1/2 \Rightarrow L(1/2)$
- $\alpha = 1$, $\beta = 2/3 \quad \Rightarrow \quad L(2/3)$
- $\alpha = 2/3$, $\beta = 1/3 \quad \Rightarrow \quad L(1/3)$

$L(1/3)$

Function field sieve for $\mathbb{F}_{2^g}$

$$\mathcal{C} : Y^d = F(X), \quad F \in \mathbb{F}_2[X], \deg F \approx d$$

$$\begin{tikzcd} \mathbb{F}_2[\mathcal{C}] = \mathbb{F}_2[X, Y]/(Y^d - F) \arrow[r, "Y \mapsto m(X)"] \arrow[d] & \mathbb{F}_2[X] \arrow[d] \\ \mathbb{F}_2[C]/\mathfrak{f} \arrow[r, phantom, "\simeq"] & \mathbb{F}_2[X]/(f) \end{tikzcd}$$

$$\mathfrak{f} = (f(X), Y - m(X)) \text{ with } f | m^d - F$$

$$\begin{array}{ccc} a(X) + b(X)Y & \mapsto & a(X) + b(X)m(X) \\ || & & || \\ \prod \mathfrak{p}_j^{f_j} & & \prod p_i^{e_i} \end{array}$$

$$\sum f_j \log(\mathfrak{p}_j) = \sum e_i \log p_i$$

Complexity of the function field sieve

- $d = g^\delta$
- $\deg m = g/d = g^{1-\delta}$
- $\deg a$, $\deg b = g^\delta$
- rational sieve

$$\begin{aligned} \deg(a + bm) &= \max(\deg a, \deg b + \deg m) \\ &= g^\gamma + g^{1-\delta} \\ &= g^{\max(\gamma, 1-\delta)} \end{aligned}$$

- algebraic sieve

$$\begin{aligned} \mathrm{N}_{\mathbb{F}_2[\mathcal{C}]/\mathbb{F}_2[X]}(a + bY) &= (-a)^d - b^d F \\ \deg \operatorname{div}(a + bY) &= \deg \mathrm{N} \\ &= d \deg b + \deg F \\ &= g^{\delta+\gamma} \end{aligned}$$

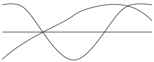
$$\gamma = \delta = 1/3 \Rightarrow g^{2/3} \text{ for both}$$

Lectures from the function field sieve

- $L_{2^g}(1/3)$ for the finite field $\mathbb{F}_{2^g}$
- uses a curve over the base field $\mathbb{F}_2$
  - double representation of $\mathbb{F}_{2^g}$ as residual field, rational and algebraic
  - degree d of the curve gives additional degree of freedom
- We already have a curve!

$L(1/3)$

$C_{n,d}$ curves



- $Y^n + h(X, Y) = X^d$
  terms in h of the form $X^i Y^j$ with $ni + dj < nd$
- unique place at infinity
- $g = \frac{(n-1)(d-1)}{2} \approx n \cdot d$
- $n \approx g^{1/3}$, $d \approx g^{2/3}$

Relations = divisors of polynomials

$$\mathcal{C} : Y^n + h(X, Y) = X^d$$

- $n \approx g^{1/3}$, $d \approx g^{2/3}$
- $\varphi = a(X)Y + b(X)$
- $\deg a$, $\deg b \approx g^{1/3}$
- $\#$ affine zeroes $= \deg_X \mathrm{N}_{\mathbb{F}_q[\mathcal{C}]/\mathbb{F}_q[X]}(\varphi)$
  - $\mathrm{N}(\varphi) = \mathrm{Res}_Y(\varphi, \mathcal{C})$
  - $\deg \mathrm{N}(\varphi) \leq \deg_X \varphi \cdot \deg_Y \mathcal{C} + \deg_Y \varphi \cdot \deg_X \mathcal{C} \approx 2g^{2/3}$
- affine divisor of degree $g^{2/3}$
- sum of prime divisors of degree $g^{1/3}$ with probability $1/L(1/3)$
  $\Rightarrow$ relation

Related work

- Adleman–DeMarrais–Huang 1994, applied to a special class of curves
- for small genus, essentially Diem 2006

Why not $L(1/4)$?

$$\mathcal{C} : Y^n + h(X, Y) = X^d$$

- $n \approx g^{1/2}$, $d \approx g^{1/2}$
- $\varphi = a(X)Y + b(X)$
- $\deg a$, $\deg b \approx g^0$
- $\#$ affine zeroes $= \deg_X \mathrm{N}_{\mathbb{F}_q[\mathcal{C}]/\mathbb{F}_q[X]}(\varphi)$
  - $\mathrm{N}(\varphi) = \mathrm{Res}_Y(\varphi, \mathcal{C})$
  - $\deg \mathrm{N}(\varphi) \leq \deg_X \varphi \cdot \deg_Y \mathcal{C} + \deg_Y \varphi \cdot \deg_X \mathcal{C} \approx 2g^{1/2}$
- affine divisor of degree $g^{1/2}$
- sum of prime divisors of degree $g^{1/4}$ with probability $1/L(1/4)$

Therefore not $L(1/4)$!

- size of the search space:

$$\underbrace{q^{g^{1/3}}}_{\#a} \cdot \underbrace{q^{g^{1/3}}}_{\#b} = q^{2g^{1/3}} \approx L(1/3) = \#\mathit{trials}$$

Application: Group structure computation

- class of curves $\mathcal{C} : Y^n + h(X, Y)$ with h of degree d in X, $< n$ in Y
  - not necessarily $C_{n,d}$
  - $n \leq n_0\, g^{1/3}\, \mathcal{M}^{-1/3}$
  - $d \leq d_0\, g^{2/3}\, \mathcal{M}^{1/3}$
  - $\mathcal{M} = \frac{\log(g \log q)}{\log q}$
  - $g > \log^{2+\varepsilon} q$
- algorithm
  - Compute an approximation to $h = \#J(\mathcal{C})$ within a factor of 2.
  - Fix smoothness bound $B = \log_q L_{q^g}(1/3, \rho)$.
  - Enumerate factor base $\mathcal{F}$.
  - Fill a matrix of size $L_{q^g}(1/3, \rho)$ with relations.
  - Compute the Smith normal form of the matrix.
  - Return h and generators of the group.
- running time depends on n_0 and d_0
  $L_{q^g}(1/3, > 25/24)$

1. Discrete logarithms in $L(1/2)$
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What is the problem?

$$\mathcal{C} : Y^n + h(X, Y) = X^d$$

- $n \approx g^{1/3}$, $d \approx g^{2/3}$
- Given $E = xD$, find x.
- need relations with D and E (or a linear combination of them)
- problem: fixed divisors of degree $g = g^1$
- broken into pieces of degree $g^{2/3}$ in time $L(1/3)$ (construction kit lemma)

Special Q-sieve

$$Q = \mathrm{div}(u, Y - v), \quad \deg u, \deg v = g^{2/3}$$

- φ going through Q
- $\varphi = au + b(Y - v) = (au - bv) + bY$
- $\deg_X \varphi \approx g^{2/3}$
- affine divisor of degree g^1
- broken into pieces of degree $g^{2/3}$ in time $L(1/3)$

Solution: Spend some more time $L(1/3 + \varepsilon)$

- divisor of degree g broken into pieces of degree $g^{2/3-\varepsilon}$ in time $L(1/3 + \varepsilon)$

special Q-sieve

- $Q = (u, Y - v)$, $\deg u$, $\deg v = g^{2/3-\varepsilon}$
- $\varphi = au + b(Y - v) = (au - bv) + bY$
- $\deg_X \varphi \approx g^{2/3-\varepsilon}$
- affine divisor of degree $g^{1-\varepsilon}$
- broken into pieces of degree $g^{2/3-2\varepsilon}$ in time $L(1/3 + \varepsilon)$

Running time

- tree of special Q
- height $\leq 1 + (1/3)/\varepsilon$
- fan-out bounded by g (on average $g^{1/3}$)
- number of nodes $\approx g^{1/(3\varepsilon)}$
  polynomial in g!

$$L_{q^g}(1/3 + \varepsilon, c + o(1))$$

What is the constant? 0!

$$L(1/3 + \varepsilon/2, c + o(1)) \subseteq L(1/3 + \varepsilon, o(1))$$

Solution: Increase the degree in Y

- $Q = (u, Y - v)$
  $= [u, Y - v_1, Y^2 - v_2, Y^3 - v_3, \ldots], \; v_i = v^i \bmod u$
- $\deg u = \deg v_i = g^\alpha$
- $\varphi = \sum_{i=1}^{k} r_i (Y^i - v_i) = -\underbrace{\sum r_i v_i}_{g^{1/3} \deg} + \underbrace{\sum r_i Y^i}_{dg^{1/3} + kg^{2/3}}$; $\deg r_i = d$
- put $d = kg^{1/3}$
- degree of $\sum r_i v_i$: $\quad d + g^\alpha$
- degrees of freedom: $\quad kd$
- need $\quad kd = g^\alpha$
- $k = g^{\alpha/2 - 1/6}$, $d = g^{\alpha/2 + 1/6}$, degree of divisor $g^{\alpha/2 + 1/2}$
- smoothing in time $L(1/3)$ towards $g^{\alpha/2 + 1/6}$

Running time

- tree of special Q
- height $\leq g^{2/3}$
- fan-out bounded by g
- number of nodes $\leq g^{4/3}$
  polynomial in g!

$$L_{q^g}(1/3, c + o(1))$$

More balanced curves

$$\mathcal{C}: Y^n + h(X, Y) = X^d$$

- $n \approx g^\alpha$, $d \approx g^{1-\alpha}$, $\alpha \in [1/3, 1/2]$
- $\varphi = a_0(X) + a_1(X)Y + \cdots + a_k(X)Y^k$
- $\deg a_i = g^{2/3-\alpha}$, $k = g^{\alpha-1/3}$
- $\deg \mathrm{N}(\varphi) \leq \deg_X \varphi \cdot \deg_Y \mathcal{C} + \deg_Y \varphi \cdot \deg_X \mathcal{C} \approx 2g^{2/3}$
- relations and group structure in $L(1/3)$
- discrete logarithms in $L(\alpha + \epsilon)$

Conclusion

First algorithm solving the discrete logarithm problem for algebraic curves in $L(1/3)$

Outlook

- further class of curves by Diem
- characterise all curves with an algorithm in $L(1/3)$?