Towards an exact cost analysis of index-calculus algorithms

ECC 2006

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Work in progress with *Roberto Avanzi*

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This is often stated as $O(\sqrt{\tilde{q}})$, hiding the constant $\frac{\sqrt{\pi}}{2}$ (assuming a group of prime order).

Genus 2 HEC DLP

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To get the same security as an elliptic curve over $\mathbb{F}_{\tilde{q}}$, we ask that q has half the number of bits as $\tilde{q}$ (abusing notation, we ask that $O(\sqrt{\tilde{q}}) = O(q)$).

We then compare the efficiency of $Jac(H)(\mathbb{F}_q)$ with $E(\mathbb{F}_{\tilde{q}})$.

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We then compare the efficiency of $Jac(H)(\mathbb{F}_q)$ with $E(\mathbb{F}_{\tilde{q}})$.

Since the security of both group depends on the same algorithm, the hidden constants are the same, so we might expect a fair comparison.

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Group Operations

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We need to take into account the difference in the cost of the group operations.

Convention: We define the average group operation as the average cost per group operation in a NAF.

Genus 1 vs Genus 2

In characteristic 2, we find

$$\approx 0.5653\sqrt{\tilde{q}}$$

for $E(\mathbb{F}_{\tilde{q}})$ of order $2 \cdot \text{prime}$ at 157 bits, and at 223 bits:

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For genus 2, we find

$$\approx 0.5283\sqrt{q}$$

for $Jac(H)(\mathbb{F}_q)$ of order $2 \cdot$ prime at 79 bits and at 109 bits:

$$\approx 0.5348\sqrt{q} \quad .$$

Genus g

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for the double large prime method (2LP), and

$$O\left(q^{2-\frac{2}{g+1/2}+\epsilon}\right)$$

for the single large prime method (1LP).

Problems

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The impact of index calculus at cryptographic sizes depends a lot on the value of the constant.

The smaller the constants are, the lower the security of $Jac(H)(\mathbb{F}_q)$ really is. If the constant is too large Pollard Rho could still be the fastest method to solve the DLP.

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Index calculus does not exclude security, but it could exclude efficiency!

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- The algorithm is more complicated
- Not all costs involved are group operations
- We haven't finished writing it up...

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But:

- Even rough estimates give a bigger constant for 2LP.
- The analysis is tighter for 1LP.
- For genus ≥ 4, 1LP could win.
- At cryptographic sizes, some of the approximations used to obtain the asymptotic form have a big impact on the constant.
- One improvement does not apply to 2LP (so far).

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which gives, for a factor base of size B:

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All other parts of the algorithm can be put in the $+\epsilon$ (so we ignore them).

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After playing around, we obtain a minimum of

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To improve the running time, we can:

1. Improve c_L
2. Improve c_W
3. Introduce a new factor

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Linear Algebra

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- Berlekamp–Massey can be done in sub-quadratic time, so we ignore that cost (for now).
- To reduce c_L, we need block Wiedemann (in progress).

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- Very limited impact for HEC index calculus.
- In progress...

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The result: Harvesting!

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$$T_k(B) = c_W \cdot 3\sqrt{2}\sqrt{k}B^{-5/2}q^{7/2} + c_L \cdot \frac{3}{2}p(k)^2B^2$$

and the minimal value is

$$T_{k,min} = 3\left(\frac{2^{7/9}}{5^{5/9}} + \frac{5^{4/9}}{2^{11/9}}\right){c_L}^{5/9}{c_W}^{4/9}\left(kp(k)^5\right)^{2/9}q^{14/9}$$

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The good news: $f(k) = kp(k)^5$ is strictly decreasing.
The bad news: hard to analyze and decreases slowly.

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Harvesting

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For $q = 2^{53}$, we have some interference due to the field size, and we need to increase k by 3.29% in the random walk side, giving us

$$\tilde{f}(4)^{2/9} = (4 \cdot 1.0329 \cdot p(4)^5)^{2/9} \approx 0.7142$$

For $q = 2^{73}$, the interference is close to 0, and we are very close to

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We get a time-memory trade-off.
Harvesting has a similar impact on 0LP, but we don't know how to predict the impact on 2LP.

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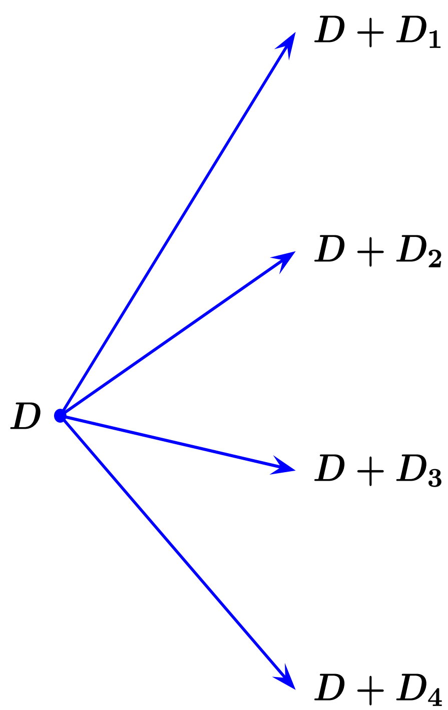
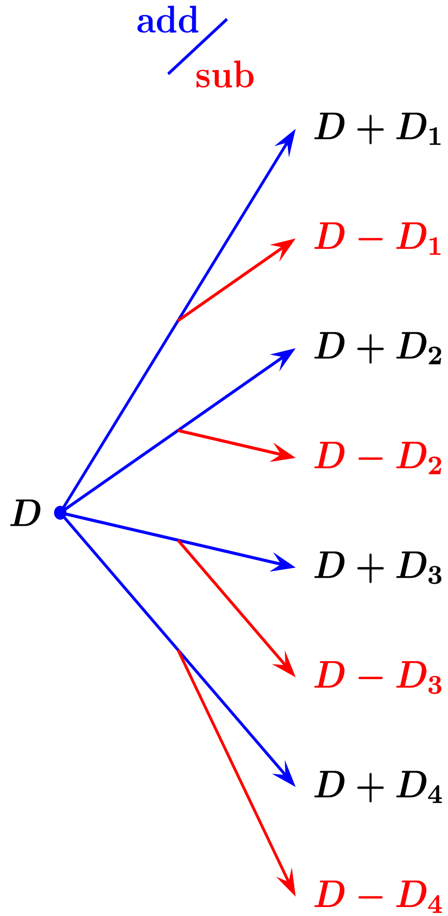
Point 2 and 3 have a huge impact on c_W, and they depend on algorithms to factor polynomials.

D

add

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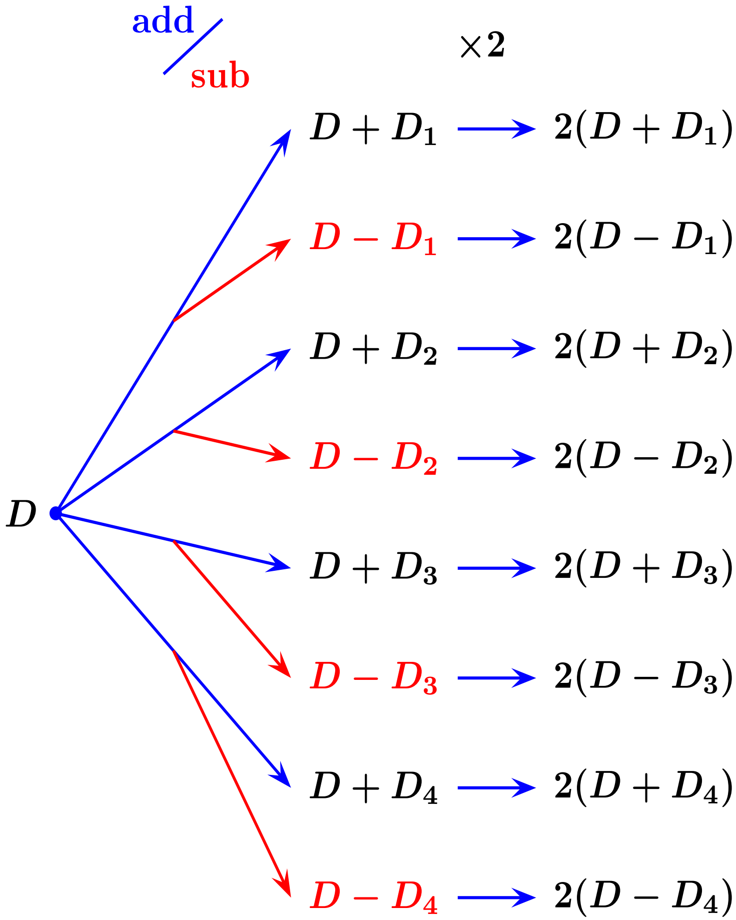
Random Walk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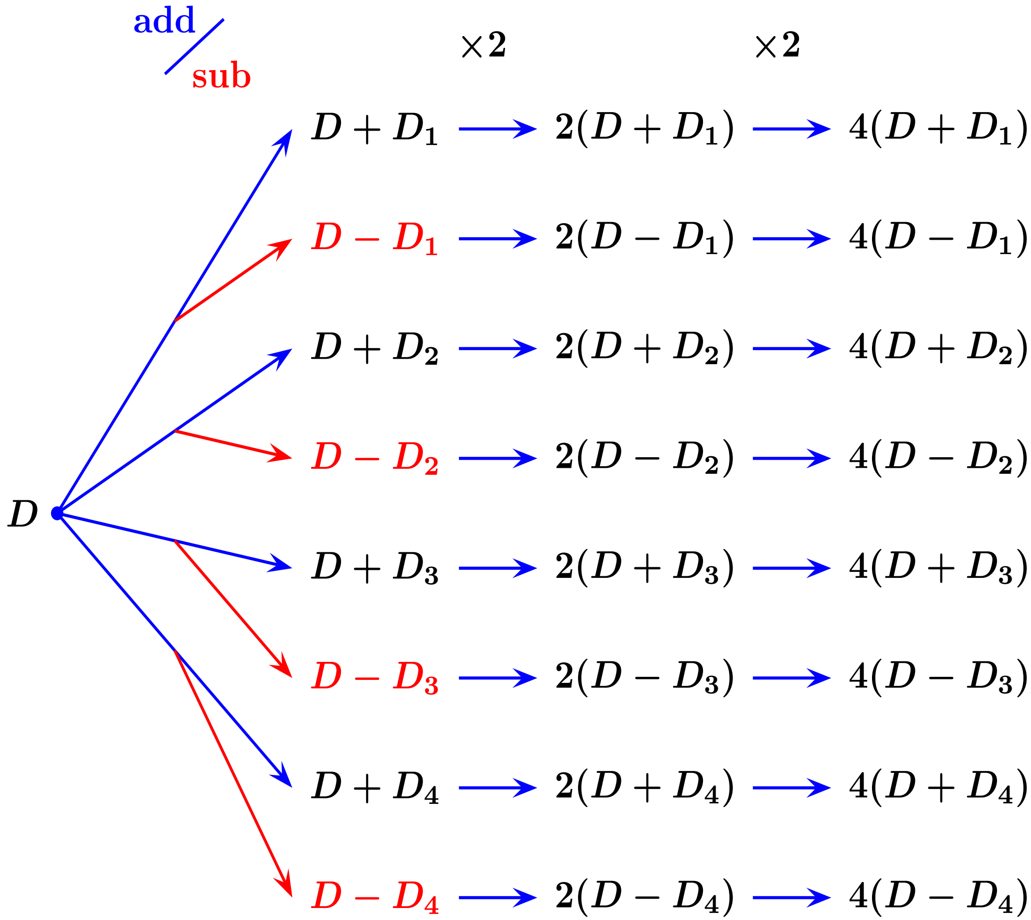
Random Walk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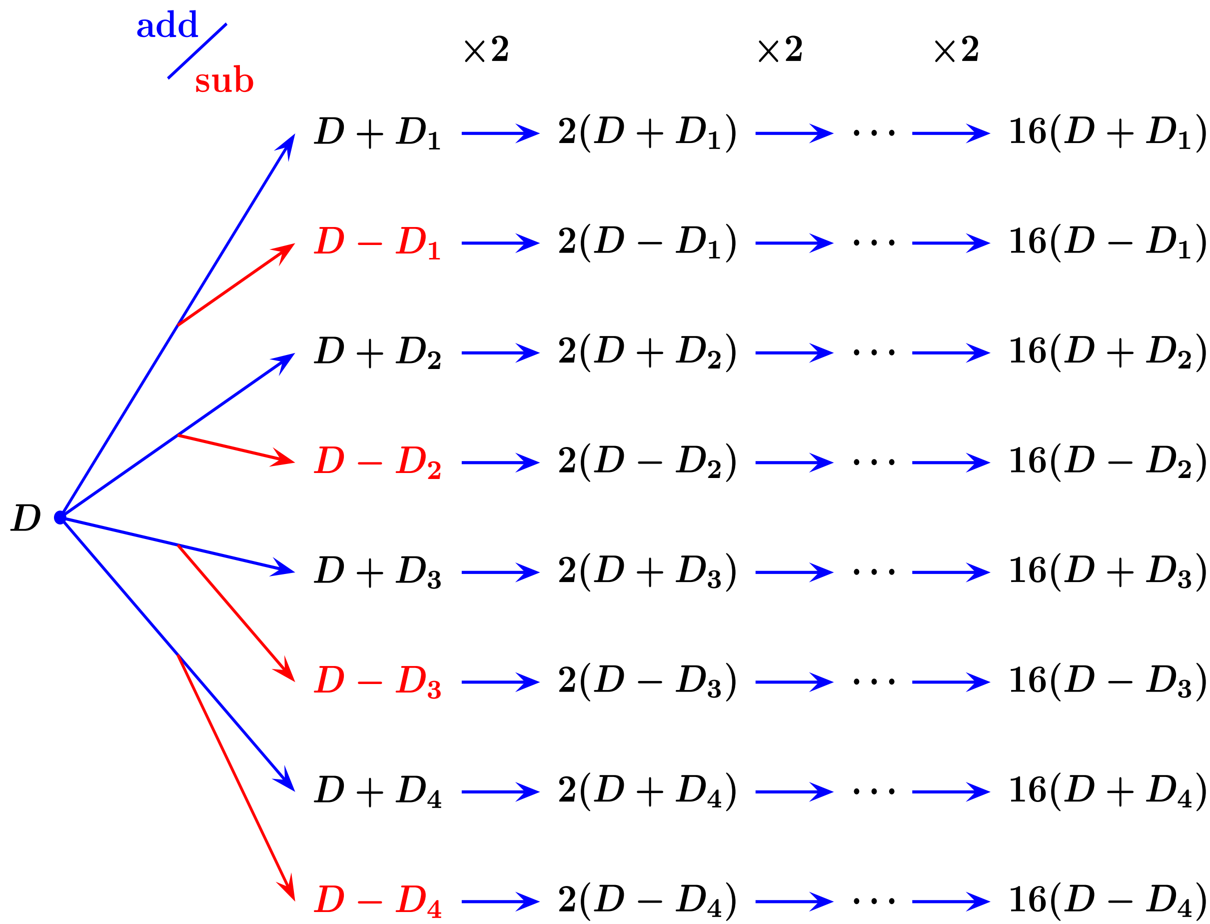
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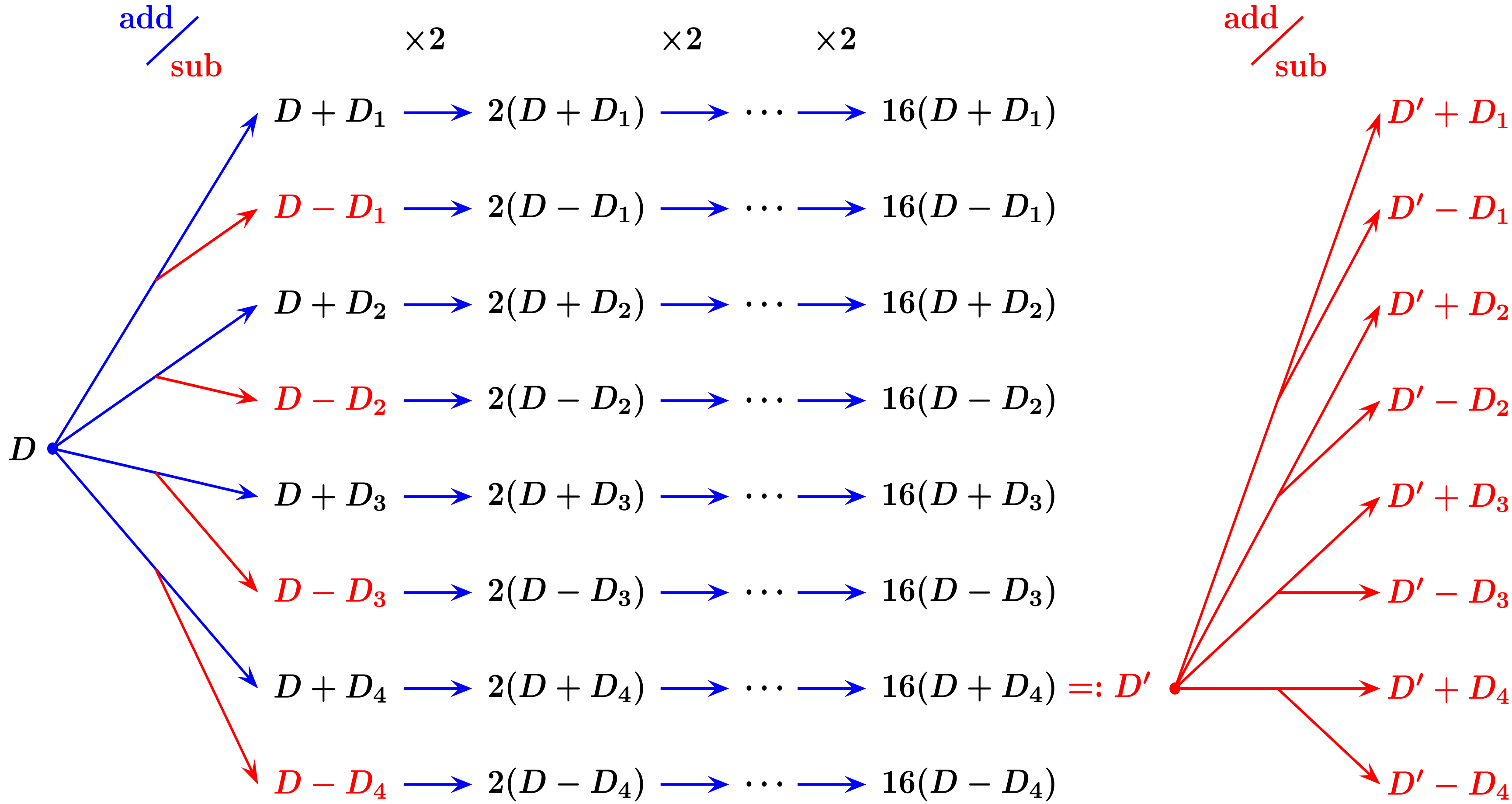
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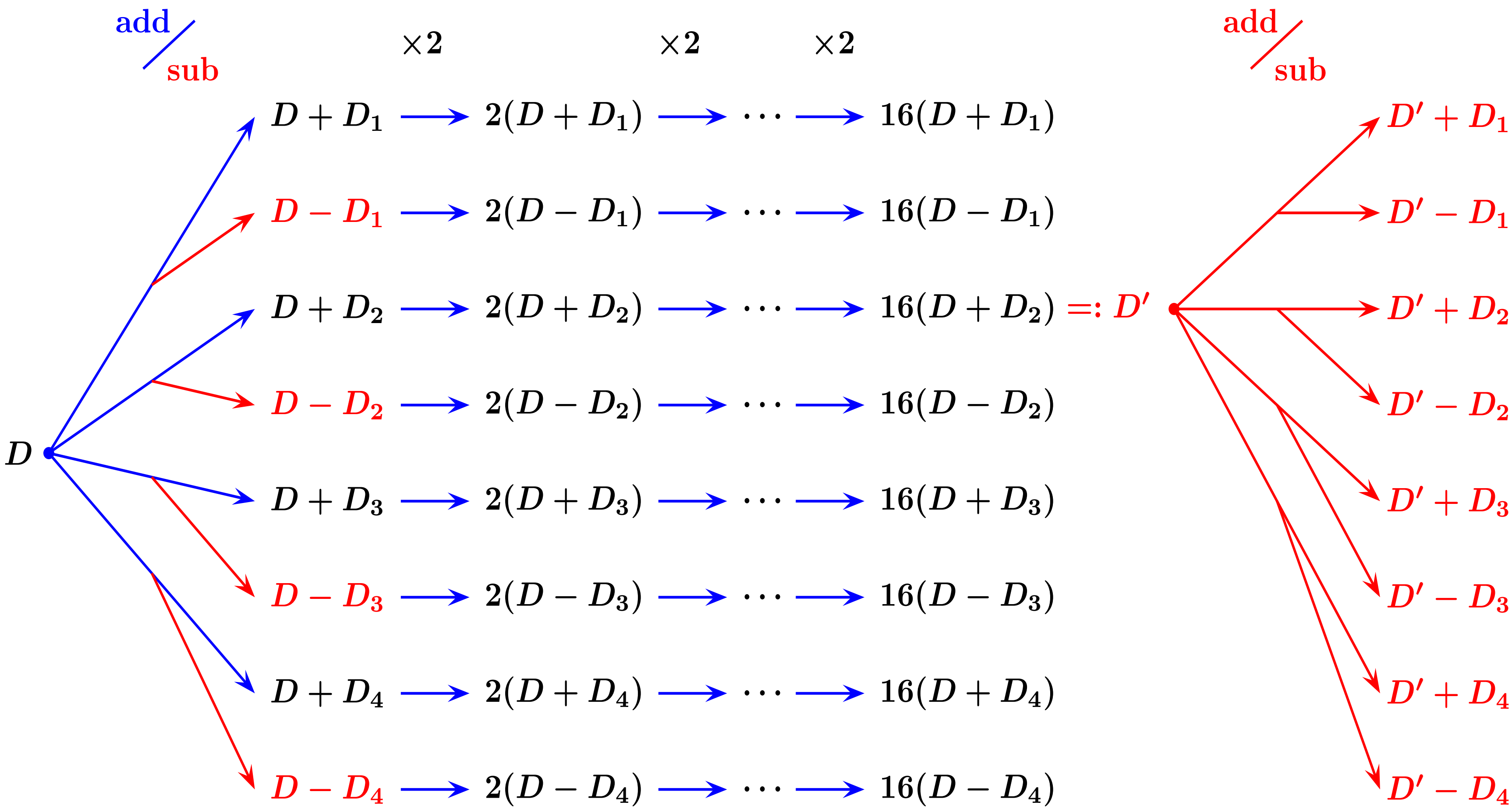
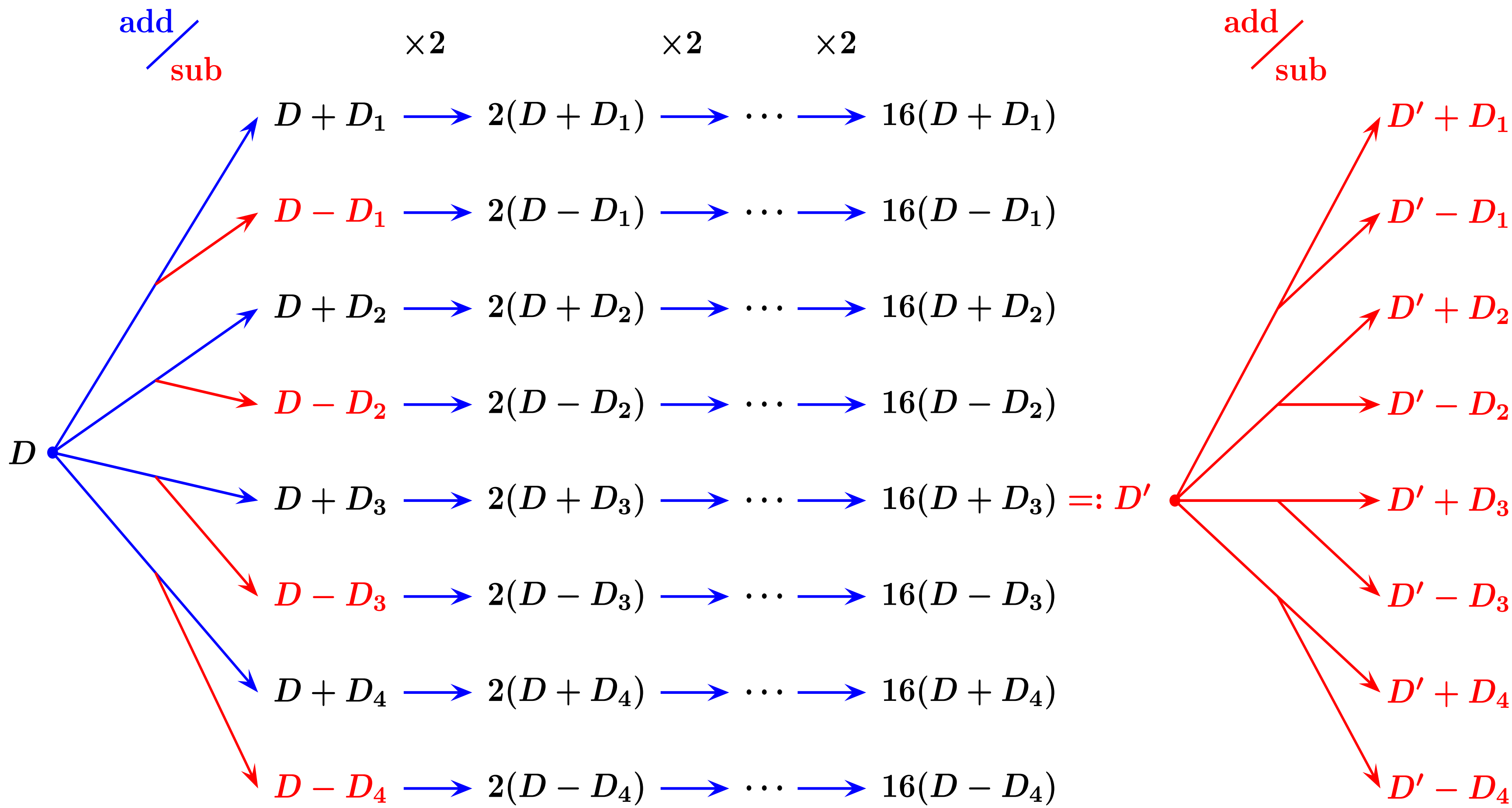
$D' - D_4$

Random Walk


add
sub
×2
×2
×2
D
D + D1 → 2(D + D1) → ··· → 16(D + D1)
D − D1 → 2(D − D1) → ··· → 16(D − D1)
D + D2 → 2(D + D2) → ··· → 16(D + D2) =: D′
D − D2 → 2(D − D2) → ··· → 16(D − D2)
D + D3 → 2(D + D3) → ··· → 16(D + D3)
D − D3 → 2(D − D3) → ··· → 16(D − D3)
D + D4 → 2(D + D4) → ··· → 16(D + D4)
D − D4 → 2(D − D4) → ··· → 16(D − D4)
add
sub
D′ + D1
D′ − D1
D′ + D2
D′ − D2
D′ + D3
D′ − D3
D′ + D4
D′ − D4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D
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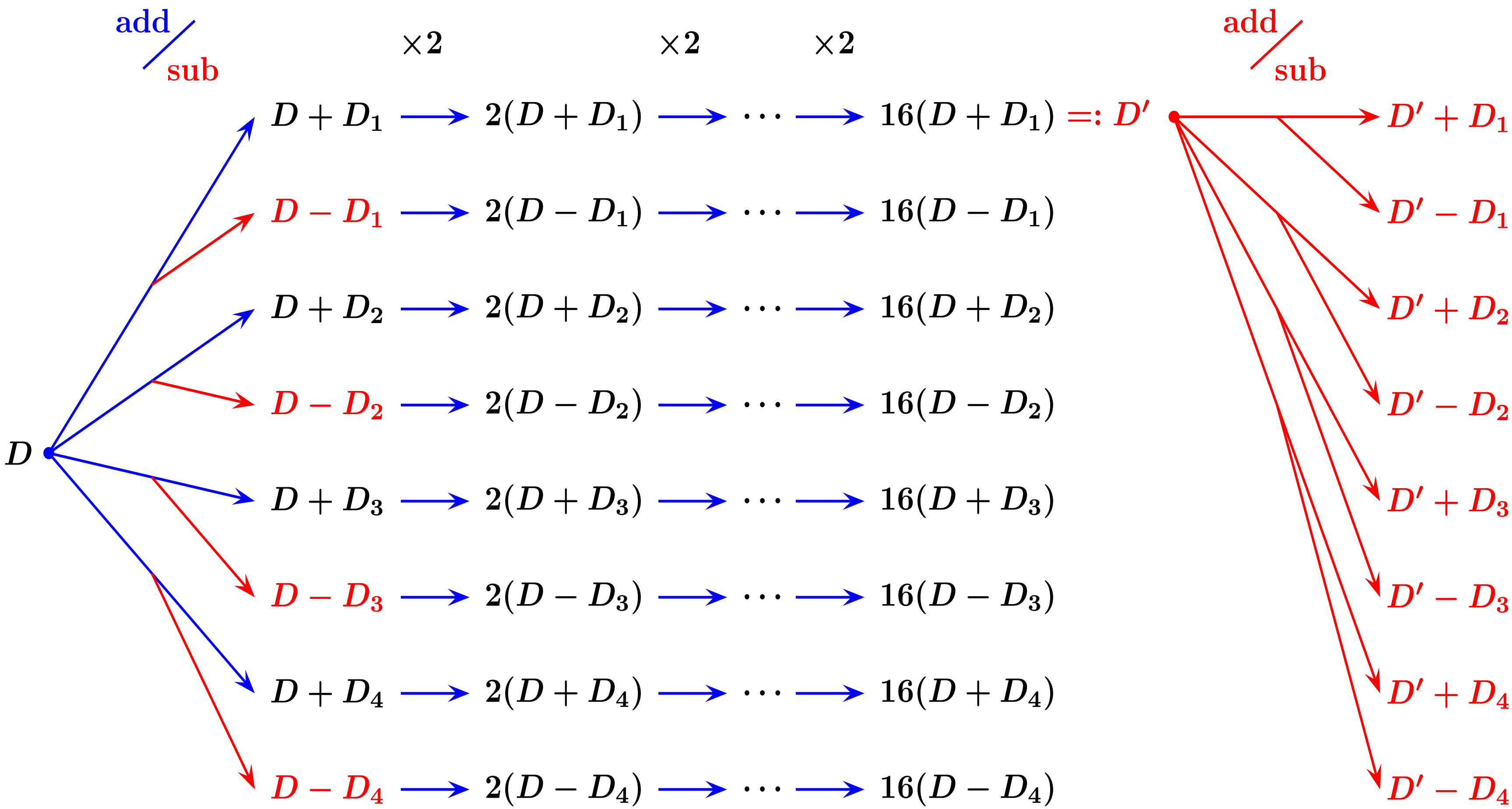

Courtesy of Roberto Avanzi

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- We can use Montgomery's trick for the inversions (and adapt our group operations).

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  - Choosing u_i irreducible minimizes the risk of having $\gcd(u, u_i) \neq 1$.
  - It takes $\approx 3q$ steps of pre-random-walk to find each D_i (goes into the $+\epsilon$).

Given a polynomial of small degree d over $\mathbb{F}_q$, the standard factorization algorithms take $O(d^2 \log_2(q))$ field operations to test if it splits, and to factor it into linear terms when possible.

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After several hours of fun, and many pages of operation counts and implementation (finding ratios between the different operations), we found

Field size	step of random walk	average group op.
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Combining everything to get c_T in $T_{4,min} = c_T q^{14/9}$, we find

$$c_T(53) \approx 2.105 \qquad c_T(73) \approx 1.547$$

If we try to apply the same approach to 2LP (without the filter), we find for $c_{\tilde{T}}$ in $\tilde{T}_{min} = c_{\tilde{T}} q^{3/2}$:

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Ignoring the constants leads us to underestimate the security of hyperelliptic curve of genus four by a few bits.

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Note: This estimates suffers from similar issues as those for 2LP in genus 4, on top of which the computations have to be checked...