Cryptanalysis of a Public Key Cryptosystem Using Matrices Over a Ring

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Abstract. At ACISP 2000, Yoo *et al* proposed a fast public key cryptosystem using matrices over a ring. The authors claim that the security of their system is based on the RSA problem. In this paper we present a heuristic attack that enables us to recover the private key from the public key. In particular, we show that breaking the system can be reduced to finding a short vector in a lattice which can be achieved using L^3-lattice reduction algorithm.

Key words: public key cryptography, cryptanalysis, L^3-algorithm

1 Introduction

Most practical public key schemes are very slow compared to symmetric key schemes. This motivates extensive research for a faster public key schemes. Among these schemes were several lattice-based systems such as [1], [2]. Both of theses schemes, which were based on the closest vector problem and the shortest vector problem, were broken [5] [6]. In fact, the L^3 algorithm was successfully used to attack many similar public key systems [4]. Yoo *et al* [8] proposed a fast public key cryptosystem similar to the system proposed in [2]. However, they claim that since the security of their scheme is based on the RSA and not the lattice problems, their scheme is secure against these lattice basis reduction attacks. In this paper we show that breaking this system is equivalent to the problem of finding a short vector in a lattice which can be solved using the L^3-lattice reduction algorithm [3]. In particular, our heuristic attack enables us to recover the private key from the public key and hence represent a total break for the proposed system.
The paper is organized as follows. In section 2 we give a description for the system. In section 3, we describe our attack. Finally we give a numerical example using the same parameters of the encryption-decryption example in [8].

2 Description of the Proposed Scheme

In this section we review the proposed public key scheme. Further details and justification for the bounds on the parameters can be found in [8].

Let n be the dimension of a lattice. The basic steps to choose the parameters are as follows:

1 Choose a positive integers m, e, d_{ii}, $1 \leq i \leq n$, primes p, q and matrix $D \in Mat_n(\mathbb{Z})$ with the following conditions:

1.1 $N = pq$.

1.2 m, e random integers such that $m \approx q^{0.4}, e \approx q^{0.3}$, where m and e are upper bounds of messages and error vectors respectively.

1.3 D : diagonal matrix such that $m < |d_{ii}| < q^{0.5}$, where $d_{ii}, 1 \leq i \leq n$ are diagonal entries of D.

2 Choose an invertible matrix $T = (t_{ij})_{1 \leq i,j \leq n} \in Mat_n(\mathbb{Z})$ such that $\sum_{j=1}^{n} t_{ij} < q^{0.2}$.

3 Form the matrices $R = DT$ and $B = B_q UL \text{ mod } N$ where $B_q = R^{-1} \text{ mod } q$, L (respectively U) are uni-modular lower (respectively upper) triangular matrix whose all entries except the diagonal entries are multiples of q.

B, e, m and N are public information. R, q and T are kept secret.

Encryption: Let $M = (m_1, \cdots, m_n)^t, 0 \leq m_i < m$ be a message vector and $E = (e_1, \cdots e_n)^t, 0 \leq e_i < e$ be an arbitrary error vector. Then the cipher text is

$$C = (BM + E) \text{ mod } N$$

Decryption: At first compute $X = (x_1, \cdots, x_n)^t$:

$$C_q = C \text{ mod } q$$

$$X = RC_q \text{ mod } q$$

Then $m_i = x_i (\text{mod } d_{ii})_{1 \leq i \leq n}$.

3 Attacking the scheme

In this section we will present a heuristic attack that enables us to recover the private key from the public key. In particular, this attack enables us to factor N using the matrix B only. As mentioned in [8], once q is revealed, one can find B_q and D and the system is totally broken. Recall that

$$B^{-1} \text{mod } q = R.$$

The following lemma follows by noting that for $N = pq$ and for any integer a we have $a \text{ mod } q = (a \text{ mod } N) \text{ mod } q$. Let $V = B^{-1} \text{mod } N$. Then we have

Lemma 1.

$$V \ mod\ q = (B^{-1} \ mod\ N) \ mod\ q = (B^{-1} \ mod\ q) = R.$$

Let r_{max} denote $\max_{\{i,j\}} |r_{ij}|$. Then from Section 3 in [8] we have

$$r_{max} < \frac{q-m}{ne} \approx \frac{q-q^{0.4}}{nq^{0.3}} < q^{0.7}.$$

Thus every elements of the matrix V can be represented as

$$v_{ij} = a_{ij}q + r_{ij},$$

where $0 \le a_{ij} < q, r_{ij} < r_{max}, 1 \le i,j \le n$.

The basic steps in the attack are as follows:

1. Calculate the matrix $V = B^{-1} \bmod N$.

2. Pick an $m, m \le n^2$, elements from the set $\{v_{ij}\}_{\{1\le i,j\le n\}}$. Let $S = \{s_i\}_{\{1\le i\le m\}}$ denote the set formed from the elements above.

3. Use the L^3 algorithm to find a reduced basis B for the $(m+1)$-dimensional lattice L which is generated by the rows of the matrix

$$\begin{bmatrix} N & 0 & 0 & \cdots & 0 & 0 \\ 0 & N & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & N & 0 \\ -s_1 & -s_2 & -s_3 & \cdots & -s_m & 1 \end{bmatrix}.$$

4. For each row $l = (l_1, l_2, \cdots, l_m, l_{m+1})$ in B such that $l_{m+1} \ne N$ do the following:

- Evaluate $gcd(N, l_{m+1})$.

- If $gcd(N, l_{m+1}) \ne 1$, return $p = gcd(N, l_{m+1})$.

5. Return (Failure).

Lemma 2. *The vector*

$$x = ((a_1N - ps_1), (a_2N - ps_2), \cdots, (a_mN - ps_m), p) = (-p\delta_1, -p\delta_2, \cdots, -p\delta_m, p)$$

is in L and has length less than approximately $(\sqrt{m+1}\ pq^{0.7})$.

Proof. The first part follows by noting that x is a linear combination of the rows of L. The second part follows by noting that each of the elements s_i can be represented as $s_i = a_iq + \delta_i$ where $\delta_i < q^{0.7}$.

From the lemma above, x is short compared to the original basis vectors, which are of length roughly N. Hence there is a good possibility that the L^3 algorithm will produce a reduced basis which include the vector x. If no solution exists then we can try another subset of elements $\{v_{ij}\}$. Our experimental results shows that the L^3 algorithm finds p with high probability.

Let $\{b_1, b_2, \cdots, b_{m+1}\}$ denote the basis of the lattice L above. Let $C \in \mathbb{R}$ be such that $|b_i|^2 \le C$ for $i = 1, 2, \cdots m+1$ and $|b_i|$ denote the length of the basis b_i. From [3], the number of arithmetic operations needed by the L^3 algorithm is $O((m+1)^4 logC)$, on integers of size $O((m+1)logC)$.

4 Numerical Example

In order to illustrate the steps in our cryptanalysis, we will use the same numerical example given in [8]. Let $q = 10570841$ and $p = 10570837$. Then $N = 111742637163917$. Let

$$D = \begin{bmatrix} 612 & 0 & 0 & 0 \\ 0 & 681 & 0 & 0 \\ 0 & 0 & 697 & 0 \\ 0 & 0 & 0 & 601 \end{bmatrix},$$

and

$$T = \begin{bmatrix} 5 & 2 & 3 & 7 \\ 4 & 3 & 1 & 2 \\ 4 & 7 & 1 & 3 \\ 2 & 3 & 4 & 9 \end{bmatrix}.$$

Then

$$R = \begin{bmatrix} 3060 & 1224 & 1836 & 4284 \\ 2724 & 2043 & 681 & 1362 \\ 2788 & 4879 & 697 & 2091 \\ 1202 & 1803 & 2404 & 5409 \end{bmatrix}.$$

Choose

$$U = \begin{bmatrix} 1 & -10570841 & 10570841 & -10570841 \\ 0 & 1 & 10570841 & -10570841 \\ 0 & 0 & 1 & 10570841 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 10570841 & 1 & 0 & 0 \\ 10570841 & -10570841 & 1 & 0 \\ -10570841 & 10570841 & 10570841 & 1 \end{bmatrix}.$$

$$B = \begin{bmatrix} 85902782524529 & 7783949494261 & 108645955098741 & 62082137341722 \\ 37207086894442 & 97811933363455 & 31492859166426 & 47829503460547 \\ 43940929239657 & 99629428908384 & 64015171957907 & 95852228892018 \\ 100737337377789 & 6871742549039 & 58298211039553 & 15913440226477 \end{bmatrix}.$$

Since B and N are public information, we can calculate

$$V = B^{-1} \bmod N = \begin{bmatrix} 72960716256453 & 4761772750607 & 47819503708674 & 64505037116731 \\ 18354764339802 & 34264334590284 & 25746128923461 & 46666277809305 \\ 28770435964827 & 105706232411633 & 39730135919762 & 9119580812042 \\ 89276407646137 & 79398453561765 & 94718657144415 & 99534468035995 \end{bmatrix}$$

Then we arbitrarily select the set

$$S = \{v_{11}, v_{12}, v_{13}, v_{14}\} = \{72960716256453, 4761772750607, 47819503708674, 64505037116731\}.$$

Using the L^3 algorithm (See algorithm 3.101 in [3], [7]). The basis to be reduced is:

$$\begin{bmatrix} 111742637163917 & 0 & 0 & 0 & 0 \\ 0 & 111742637163917 & 0 & 0 & 0 \\ 0 & 0 & 111742637163917 & 0 & 0 \\ 0 & 0 & 0 & 111742637163917 & 0 \\ -72960716256453 & -4761772750607 & -47819503708674 & -64505037116731 & 1 \end{bmatrix}.$$

The L^3-reduced basis is:

$$\begin{bmatrix} -32346761220 & -12938704488 & -19408056732 & -45285465708 & \mathbf{10570837} \\ -87078711029 & 39709857984 & 7883945327 & 11690435622 & 4385339758 \\ -12420733475 & -4968293390 & -7452440085 & -17389026865 & 182590067501 \\ -102740951106 & -253999460687 & 146924464771 & 79909317394 & 26579009212 \\ 1917450399 & -58848334744 & -420915726704 & 231779925047 & 78377734153 \end{bmatrix}.$$

Hence we get $p = 10570837$. Once p is revealed we calculate $q = N/p$. Then we get $R = V^{-1}$ mod q. After this we calculate $d_{ii} = gcd(r_{i1}, r_{i2}, \cdots r_{in}), 1 \leq i \leq n$.

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