ON ASYMPTOTIC OPTIMALITY OF BINARY SEQUENCE FAMILIES

Nam Yul Yu and Guang Gong

Department of Electrical and Computer Engineering, University of Waterloo
200 University Avenue West, Waterloo, Ontario, Canada N2L 3G1

ABSTRACT

In this paper, lower bounds - Welch, Sidelnikov, and Levenshtein bounds - on maximum correlation of a binary sequence family of size $M = L^l$ with a positive real value l are studied by giving their asymptotic behaviors. As a result, the asymptotic bounds for binary sequence families of various family sizes are presented. Then, asymptotic parameters of maximum correlation and family size for several known binary sequence families are compared with the asymptotic bounds. From the comparison, asymptotic optimality of binary sequence families is discussed in terms of both maximum correlation and family size.

1. INTRODUCTION

On maximum correlation of sequences, Welch [14] presented its lower bound by investigating even moments of the correlation. Sidelnikov [12] also derived another lower bound from a ratio of successive even moments of the correlation. In [5], Levenshtein presented a lower bound on correlation of sequences, which is known to be the best of all known bounds [3]. All of the three bounds have been used to study optimality of codes and sequence families [3].

For code-division multiple access (CDMA) communication systems, a lot of binary sequence families with good correlation have been presented. The Gold [2] and the Kasami (small set) [4] sequences are traditionally well-known binary sequence families each of which asymptotically achieves the Sidelnikov and the Welch bounds on maximum correlation, respectively. In [9], Olsen, Scholtz, and Welch gave the bent-function sequences which also achieve the asymptotic Welch bound on maximum correlation. Boztas and Kumar [1] presented the Gold-like sequence family which has the same correlation with the Gold sequences but larger linear span. Similarly, Udaya [13] gave another binary sequence family with low correlation and large linear span. In [15], Yu and Gong generalized both of the works by providing several binary sequence families with large linear span and large family size. From Z_4-linear codes, Shanbhag, Kumar, and Helleseth [11] also presented several binary sequence families with large family size.

With respect to maximum correlation, a few binary sequence families are indeed optimum in the sense that their maximum correlations asymptotically achieve the above lower bounds. However, few efforts are made in literatures on asymptotic optimality of binary sequence families by considering maximum correlation and family size simultaneously. The efforts are significant for binary sequence families applied for CDMA communication systems because a binary sequence family with larger family size is preferred if the same maximum correlation is given.

In this work, we first study the lower bounds - Welch, Sidelnikov, and Levenshtein bounds - on maximum correlation of a binary sequence family of size $M = L^l$ with a positive real value l by giving their asymptotic behaviors. As a result, the asymptotic bounds for binary sequence families of various family sizes are presented. Then, we give the asymptotic parameters of maximum correlation and family size of several known binary sequence families and compare them with the asymptotic bounds. From the comparison, we investigate the asymptotic optimality of binary sequence families in terms of both maximum correlation and family size.

2. CORRELATION AND A FAMILY OF SEQUENCES

Let $\underline{\mathbf{a}} = \{a_i\}$ and $\underline{\mathbf{b}} = \{b_i\}$ be binary sequences of period L. *Correlation* of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ is defined by

$$C_{\underline{\mathbf{a}},\underline{\mathbf{b}}}(\tau) = \sum_{i=0}^{L-1} (-1)^{a_i + b_{i+\tau}}, \quad 0 \leq \tau \leq L-1$$

where τ is a phase shift of the sequence $\underline{\mathbf{b}}$ and the indices are computed modulo L.

For M cyclically distinct binary sequences of period L, i.e., $\underline{\mathbf{s}}^{(j)} = (s_0^{(j)}, \cdots, s_{L-1}^{(j)}),\ 0 \leq j < M$, let $\mathcal{S} = \{\underline{\mathbf{s}}^{(0)}, \cdots, \underline{\mathbf{s}}^{(M-1)}\}$ and

$$C_{\max} = \max |C_{\underline{\mathbf{s}}^{(i)},\underline{\mathbf{s}}^{(j)}}(\tau)| \text{ for any } 0 \leq \tau < L,\ 0 \leq i, j < M$$

where $\tau \neq 0$ if $i = j$. Clearly, $C_{\max}$ is maximum of all nontrivial auto- and crosscorrelations of the sequences in $\mathcal{S}$. The set $\mathcal{S}$ is called a $(L, M, C_{\max})$ *family of sequences*, where M is a *family size* and $C_{\max}$ is a *maximum correlation (magnitude)* of $\mathcal{S}$. Throughout this paper, we use the above notations and definitions.

3. ASYMPTOTIC LOWER BOUNDS

For asymptotic behavior of lower bounds, we assume a family size M of a binary sequence family is approximately a power of period L of the sequence, i.e., $M = L^l$ where l is a positive real value. Then, we give the asymptotic lower bounds – Welch, Sidelnikov, and Levenshtein bounds – in terms of a variable $l = \log_L M$. With the asymptotic bounds, we have a measure to compare binary sequence families in terms of both maximum correlation and family size. For the asymptotic representation, we assume L is sufficiently large.

3.1. The Welch bound

For a positive integer k, the Welch bound [14] gives an inequality of $C_{\max}^{2k}$, i.e.,

$$C_{\max}^{2k} \geq \frac{1}{ML-1} \left\{ \frac{ML^{2k+1}}{\binom{k+L-1}{k}} - L^{2k} \right\}.$$

Applying $M = L^l$, we have

$$\begin{aligned} C_{\max}^{2k} &> \frac{L^{2k}}{\binom{k+L-1}{k}} - \frac{L^{2k-1}}{M} = \frac{L^{2k}}{\prod_{i=0}^{k-1}\left(\frac{L+k-1-i}{k-i}\right)} - L^{2k-l-1} \\ &> k! \left(\frac{L^2}{L+k-1}\right)^k - L^{2k-l-1} \approx k! L^k - L^{2k-l-1} \end{aligned} \tag{1}$$

where we assume $k \ll L$. If $k > l+1$, the right-hand side of (1) may be negative, which makes a trivial bound. Since k is a positive integer, let $k = \lfloor l+1 \rfloor$. Then,

$$C_{\max}^{2k}/L^k > \left((\lfloor l+1 \rfloor)! - L^{\lfloor l+1 \rfloor - (l+1)} \right). \tag{2}$$

In order to remove the dependency on L of the right-hand side of (2), we consider t such that

$$L^x < (x+1)^t = T(x,t), \quad -1 < x \leq 0 \tag{3}$$

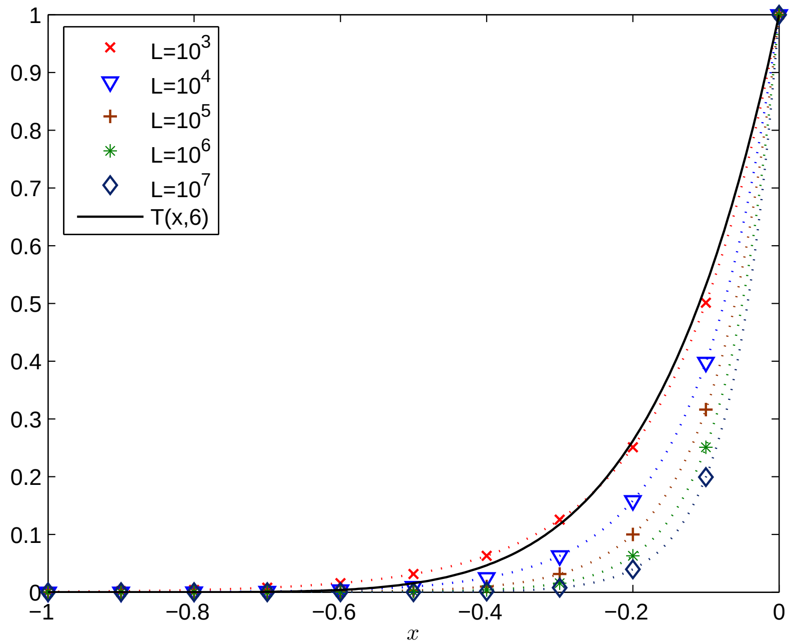

Fig. 1. $y = L^x$ and $y = T(x, 6) = (x + 1)^6$

for sufficiently large L. Fig. 1 shows $y = L^x$ and $y = T(x, t) = (x + 1)^t$ for various L's where we set $t = 6$. From Fig. 1, we see that we can choose $t = 6$ for (3) if $L > 10^3$. If we set $x = \lfloor l + 1 \rfloor - (l + 1)$, then $-1 < x \leq 0$ and from (2) and (3), we have

$$L^{\lfloor l+1 \rfloor - (l+1)} < (\lfloor l + 1 \rfloor - l)^t$$

and

$$\frac{C_{\max}}{\sqrt{L}} > \left((\lfloor \log_L M + 1 \rfloor)! - (\lfloor \log_L M + 1 \rfloor - \log_L M)^t \right)^{\frac{1}{2\lfloor \log_L M + 1 \rfloor}}. \tag{4}$$

In (4), it is clear that larger t can be chosen for more accurate asymptotic Welch bound for larger L. As a matter of fact, Welch stated the asymptotic behavior of his bound in [14]. However, (4) is more general in the sense that it contains a case of a positive real value $l = \log_L M$.

Fig. 2 shows the original and asymptotic Welch bound on normalized maximum correlation for $L = 2^n$ and $t = 6$. Note that $k = \lfloor l + 1 \rfloor$ is applied for the original Welch bound as well as its asymptotic bound. From Fig. 2, we see that the asymptotic bound is quite close to the original Welch bound for sufficiently large L.

3.2. The Sidelnikov bound

In the Sidelnikov bound [12], $C_{\max}^2$ has following inequality according to an integer k, $0 \leq k < 2L/5$.

$$\begin{aligned} C_{\max}^2 &\geq (2k + 1)(L - k) + \frac{k(k + 1)}{2} - \frac{2^k L^{2k+2}}{ML(2k)!\binom{L}{k}} \\ &= (2k + 1)L - \frac{k(3k + 1)}{2} - \frac{2^k L^{2k+2}}{ML(2k)!\binom{L}{k}}. \end{aligned} \tag{5}$$

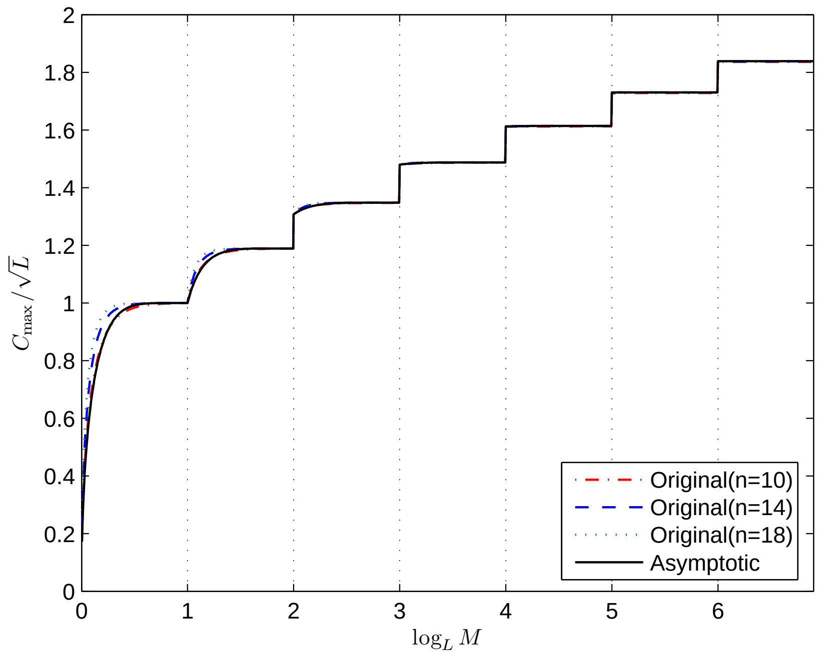

Fig. 2. Welch bound on normalized maximum correlation ($t = 6$)

Assume k is set to be sufficiently small such that $k \ll L$. Then, $\frac{k(3k+1)}{2}$ is negligible in the right-hand side of (5). With $M = L^l$, we have

$$\begin{aligned} C_{\max}^2 &\geq (2k+1)L - \frac{k!2^k L^{2k+2}}{ML(2k)!\prod_{i=0}^{k-1}(L-i)} \\ &> (2k+1)L - \frac{2^k}{\prod_{i=0}^{k-1}(2k-i)} \cdot \frac{L^{2k-l+1}}{(L-k+1)^k} \\ &> (2k+1)L - \frac{2^k}{(k+1)^k} \cdot \frac{L^{2k-l+1}}{(L-k+1)^k} \\ &= (2k+1)L - \left(\frac{2L}{(k+1)(L-k+1)}\right)^k \cdot L^{k-l+1} \\ &\approx (2k+1)L - \left(\frac{2}{k+1}\right)^k L^{k-l+1}. \end{aligned} \tag{6}$$

The last approximation is from $L \gg k$. If $k - l > 0$, the right-hand side of (6) may be negative, and then the bound becomes trivial. Since k is a positive integer, let $k = \lfloor l \rfloor$. Then,

$$C_{\max}^2/L > \left(2\lfloor l \rfloor + 1 - \left(\frac{2}{\lfloor l \rfloor + 1}\right)^{\lfloor l \rfloor} L^{\lfloor l \rfloor - l}\right) \tag{7}$$

Similar to the asymptotic Welch bound, we apply (3) to (7), and then we have

$$L^{\lfloor l \rfloor - l} < (\lfloor l \rfloor - l + 1)^t$$

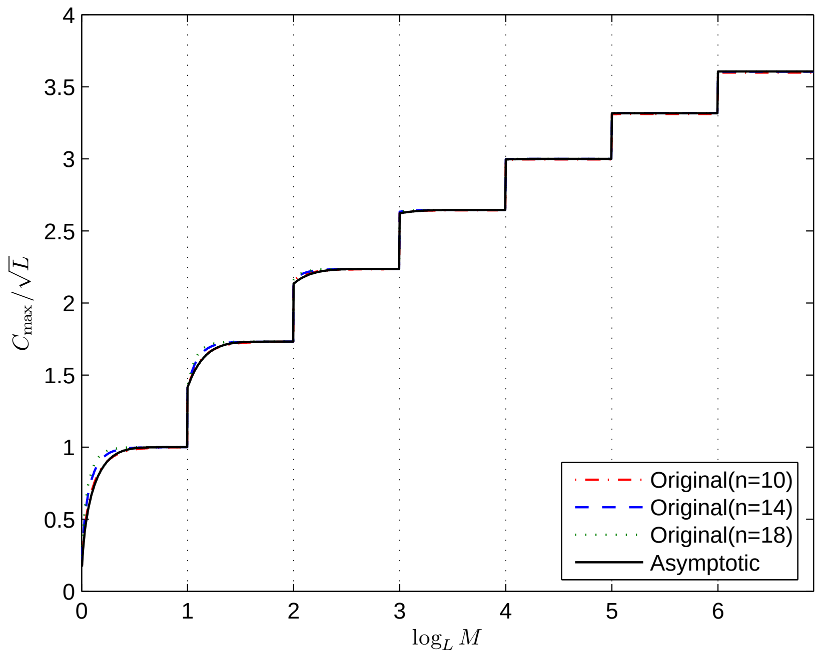

Fig. 3. Sidelnikov bound on normalized maximum correlation ($t = 6$)

for sufficiently large L, and finally

$$\frac{C_{\max}}{\sqrt{L}} > \sqrt{2\lfloor \log_L M \rfloor + 1 - \left(\frac{2}{\lfloor \log_L M \rfloor + 1}\right)^{\lfloor \log_L M \rfloor} (\lfloor \log_L M \rfloor - \log_L M + 1)^t}. \quad (8)$$

In (8), it is also clear that larger t can be chosen for more accurate asymptotic Sidelnikov bound for larger L.

Fig. 3 shows the original and asymptotic Sidelnikov bound on normalized maximum correlation for $L = 2^n$ and $t = 6$. Note that $k = \lfloor l \rfloor$ is applied for the original Sidelnikov bound as well as its asymptotic bound. Similar to the asymptotic Welch bound, we have a good asymptotic Sidelnikov bound for sufficiently large L.

3.3. The Levenshtein bound

For $M = L^l$ with a positive integer l, Levenshtein presented following asymptotic bound [6].

$$C_{\max}^2 / L > c(l)$$

where $c(l)$ is a constant of Table 1 in [6]. To derive its asymptotic bound for a positive real value l, we have to note that the Levenshtein bound coincides with the Welch bound for $0 \leq l < 1$, and also coincides with the Sidelnikov bound for $1 \leq l < 2$ [6] [3]. For $0 \leq l < 2$, therefore, we may use the asymptotic Welch and Sidelnikov bound given by (4) and (8), respectively. If $l \geq 2$, on the other hand, we simply have $C_{\max}^2 / L > c(\lfloor l \rfloor)$. In fact, it looks tricky but it is always true because the bound is an increasing function [6]. Then, we have the asymptotic Levenshtein bound given by

$$C_{\max}/\sqrt{L} > \begin{cases} \text{right-hand side of } (4), & \text{if } 0 \leq \log_L M < 1 \\ \text{right-hand side of } (8), & \text{if } 1 \leq \log_L M < 2 \\ \sqrt{c(\lfloor \log_L M \rfloor)}, & \text{if } \log_L M \geq 2 \end{cases} \quad (9)$$

Fig. 4 shows the asymptotic bounds of Welch, Sidelnikov, and Levenshtein bounds. In Fig. 4, we use $t = 6$ for (4) and (8), respectively. Asymptotically, we see that the Levenshtein bound is also the best of all the three bounds.

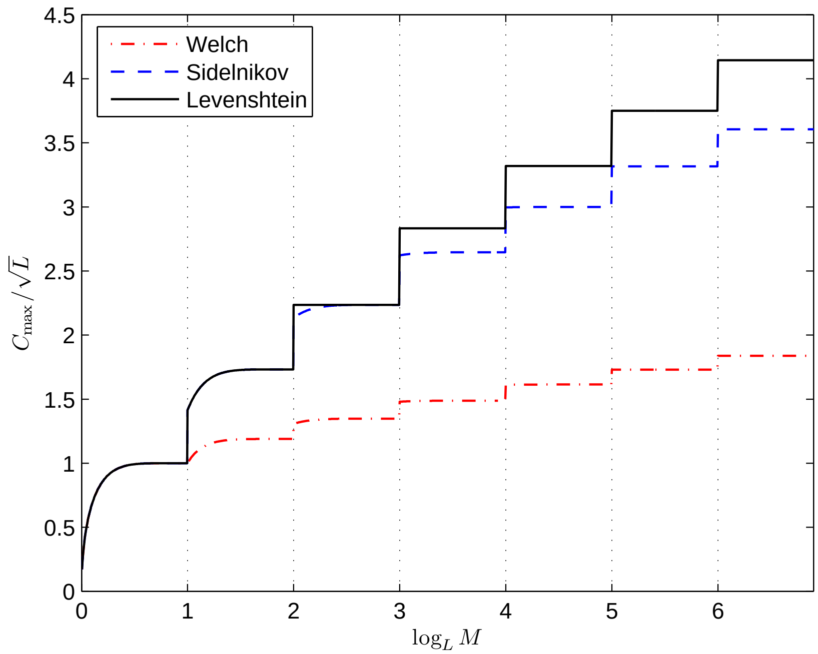

Fig. 4. Asymptotic bounds on normalized maximum correlation

4. ASYMPTOTIC PARAMETERS OF BINARY SEQUENCE FAMILIES

In this section, we list several binary sequence families by giving their asymptotic parameters – period, family size, and maximum correlation. First of all, we introduce two binary sequence families $\mathcal{S}_o(\rho)$ (or $\mathcal{S}_e(\rho)$) [15] and $\mathcal{Z}(n-1, D)$ [11], and discuss their asymptotic parameters in detail. Note that both binary sequence families have sub-sequence classes according to ρ and D, respectively. Then, we also give the asymptotic parameters of other well known binary sequence families.

4.1. $\mathcal{S}_o(\rho)$ and $\mathcal{S}_e(\rho)$

In [15], Yu and Gong presented binary sequence families $\mathcal{S}_o(\rho)$ and $\mathcal{S}_e(\rho)$ for odd and even n, respectively, which contain Gold-like ($\mathcal{S}_o(1)$) [1] and Udaya's sequences ($\mathcal{S}_e(1)$) [13] as their sub-sequence classes. According to a positive integer ρ, $1 \le \rho \le \lfloor \frac{n}{2} \rfloor$, the families have following parameters [15].

$$L = 2^n - 1, \quad M = 2^{n\rho}, \quad C_{\max} = \begin{cases} 1 + 2^{\frac{n+2\rho-1}{2}} & \text{for } \mathcal{S}_o(\rho) \\ 1 + 2^{\frac{n}{2}+\rho} & \text{for } \mathcal{S}_e(\rho) \end{cases}$$

For their asymptotic behaviors, let $L = 2^n$. Then, $\log_L M = \rho$ and $C_{\max}$ is asymptotically given by

$$C_{\max} \approx \begin{cases} 2^{\rho-1}\sqrt{2L} & \text{for } \mathcal{S}_o(\rho) \\ 2^{\rho}\sqrt{L} & \text{for } \mathcal{S}_e(\rho) \end{cases}$$

In [15], $\mathcal{S}_o(\rho)$ or $\mathcal{S}_e(\rho)$ is constructed from a linear cyclic subcode of the second order Reed-Muller code and its maximum correlation is determined by a rank of a symplectic form associated with the subcode. With respect to a rank and a symplectic form, it is known in [8] that maximum size of a set of symplectic forms with ranks of at least $2d$ is $2^{n\left(\frac{n+1}{2}-d\right)}$ for odd n. In $\mathcal{S}_o(\rho)$, therefore, we have $2d = n - 2\rho + 1$ and maximum size of the set is $2^{n\rho}$, which is identical to the family size of $\mathcal{S}_o(\rho)$. With given maximum correlation, therefore, $\mathcal{S}_o(\rho)$ has the largest family size of all binary sequences constructed from subcodes of the second order Reed-Muller code. In other words, we can say that $\mathcal{S}_o(\rho)$ is the best among all *quadratic-form* binary sequence families with respect to maximum correlation and family size.

4.2. Z_4-linear binary sequence family

A Z_4-linear binary sequence family, denoted by $\mathcal{Z}(n-1, D)$, was introduced by Shanbhag, Kumar, and Helleseth [11], where the notation is slightly changed for period $L = 2^n - 2$. For a positive integer $D \geq 2$, it has following parameters [11].

$$L = 2^n - 2, \quad M \geq \left(\frac{L}{2}\right)^{(D-\lfloor\frac{D}{4}\rfloor-1)}, \quad C_{\max} \leq \begin{cases} (D-1)\sqrt{2L}, & \text{for odd } n \\ (D-1)\sqrt{L}, & \text{for even } n \end{cases} \tag{10}$$

(In the original construction in [11], the family size was a half of M. In [3], however, it is pointed out that the size can be doubled.) In this paper, we assume that the inequalities in (10) can be replaced by equalities. From $M = \left(\frac{L}{2}\right)^{(D-\lfloor\frac{D}{4}\rfloor-1)}$, we have

$$\log_L M = \left(D - \left\lfloor\frac{D}{4}\right\rfloor - 1\right)(1 - \log_L 2) = \left(D - \left\lfloor\frac{D}{4}\right\rfloor - 1\right)\left(\frac{n-1}{n}\right)$$

where $L \approx 2^n$. Thus,

$$D - \left\lfloor\frac{D}{4}\right\rfloor - 1 = \frac{n}{n-1}\log_L M \triangleq \eta. \tag{11}$$

Since D is a positive integer, note that η is an integer while $\log_L M$ is a real value. From (11), we have

$$D = \eta + 1 + \left\lfloor\frac{\eta}{3}\right\rfloor.$$

(In some cases, we may have two different D's for the same η where we choose a smaller one for D [11].)

Finally, we have the asymptotic parameters $\log_L M$ and $C_{\max}$ represented by an integer η, i.e.,

$$\log_L M = \left(\frac{n-1}{n}\right)\eta, \quad C_{\max} = \begin{cases} \left(\eta + \lfloor\frac{\eta}{3}\rfloor\right)\sqrt{2L}, & \text{for odd } n \\ \left(\eta + \lfloor\frac{\eta}{3}\rfloor\right)\sqrt{L}, & \text{for even } n. \end{cases}$$

For simplicity, we use the notations $\mathcal{Z}_o(D)$ for odd n and $\mathcal{Z}_e(D)$ for even n, respectively, to denote the Z_4-linear binary sequence family.

For asymptotic period $L = 2^n$ of a binary sequence, Table 1 shows asymptotic parameters $\log_L M$ and $C_{\max}$ for several well known binary sequence families. In Table 1, family sizes of Rothaus and Kasami (large set) sequences are asymptotically assumed as $M = 2^{2n}$ and $2^{\frac{3n}{2}}$, respectively. Gold-like [1] and Udaya's [13] sequences are considered as sub-sequence classes of $\mathcal{S}_o(1)$ and $\mathcal{S}_e(1)$, respectively.

5. ASYMPTOTIC OPTIMALITY OF BINARY SEQUENCE FAMILIES

In this section, we compare asymptotic parameters of normalized maximum correlation and family size of binary sequence families with the asymptotic bounds in Section 3. Then we discuss the best known binary sequence family in terms of both maximum correlation and family size.

Table 1. Asymptotic parameters of several binary sequence families with given period L

Sequence	$\log_L M$	$C_{\max}$	n
Gold [2]	1	$\sqrt{2L}$	odd
Rothaus [10]	2	$2\sqrt{2L}$	odd
Kasami (small set) [4]	$1/2$	$\sqrt{L}$	even
Kasami (large set) [4]	$3/2$	$2\sqrt{L}$	even
Bent [9]	$1/2$	$\sqrt{L}$	even, multiple of 4
$\mathcal{S}_o(\rho)$	ρ	$2^{\rho-1}\sqrt{2L}$	odd
$\mathcal{S}_e(\rho)$	ρ	$2^{\rho}\sqrt{L}$	even
$\mathcal{Z}_o(D)$	$\left(\frac{n-1}{n}\right)\eta$	$\left(\eta+\left\lfloor\frac{\eta}{3}\right\rfloor\right)\sqrt{2L}$	odd
$\mathcal{Z}_e(D)$	$\left(\frac{n-1}{n}\right)\eta$	$\left(\eta+\left\lfloor\frac{\eta}{3}\right\rfloor\right)\sqrt{L}$	even

5.1. Odd n

For odd n, we consider maximum correlation normalized by $\sqrt{2L}$ for the asymptotic bounds in (4), (8), and (9) as well as for the binary sequence families, which is convenient for analysis.

Fig. 5 shows normalized maximum correlation of several binary sequence families for odd n and its asymptotic bounds ($t = 6$). Note that $\mathcal{S}_o(\rho)$ and $\mathcal{Z}_o(D)$ contain their own sub-sequence classes according to ρ and D, respectively. In Fig. 5, the Gold-like sequences (or $\mathcal{S}_o(1)$) asymptotically achieve the Sidelnikov and the Levenshtein bounds. $\mathcal{S}_o(2)$ and $\mathcal{S}_o(3)$, on the other hand, do not achieve the asymptotic bounds, but they provide larger family sizes than any other known binary sequence families with given normalized maximum correlation. The Gold and the Rothaus sequences also present the same parameters with $\mathcal{S}_o(1)$ and $\mathcal{S}_o(2)$, respectively, but they provide much smaller linear spans. $\mathcal{Z}_o(D)$, on the other hand, does not achieve the asymptotic bounds for any D in terms of normalized maximum correlation. When $D \leq 4$, however, its family size approaches to that of $\mathcal{S}_o(\rho)$ of $\rho \leq 3$, respectively, with the same normalized maximum correlation. For $D \geq 5$, $\mathcal{Z}_o(D)$ even provides better performance than $\mathcal{S}_o(\rho)$ of $\rho \geq 4$ although it is far from the asymptotic bounds.

Consequently, $\mathcal{S}_o(\rho)$ of $\rho \leq 3$ is the best known binary sequence family for odd n and $\log_L M \leq 3$ in terms of both maximum correlation and family size. With given normalized maximum correlation, it has the largest family size among all known binary sequence families and large linear span. For $\log_L M = 1$, in particular, the Gold-like sequences (or $\mathcal{S}_o(1)$) even achieve the asymptotic Sidelnikov and Levenshtein bounds. For $\log_L M \geq 4$, however, $\mathcal{S}_o(\rho)$ provides the exponentially increasing maximum correlation. In this region, $\mathcal{Z}_o(D)$ of $D \geq 5$ shows better performance than corresponding $\mathcal{S}_o(\rho)$ of $\rho \geq 4$ even though it does not achieve any asymptotic bounds.

5.2. Even n

For even n, maximum correlation normalized by $\sqrt{L}$ is considered. Fig. 6 shows normalized maximum correlation of several binary sequence families for even n and its asymptotic bounds ($t = 6$). In Fig. 6, the Kasami (small set) and the bent sequences asymptotically achieve the Welch, Sidelnikov, and Levenshtein bounds. However, their family sizes are rather small ($\sqrt{L}$). $\mathcal{Z}_e(2)$, on the other hand, not only achieves all the asymptotic bounds, but also provides family size as large as possible for sufficiently large n. When n is large, therefore, $\mathcal{Z}_e(2)$ or the *Kerdock* sequences [3] seem to be the best known binary sequences among all known binary sequence families in terms of maximum correlation and family size. $\mathcal{Z}_e(3)$ or the *Delsarte-Goethals* sequences [3], do not achieve any bounds on normalized maximum correlation, but the family size is larger than those of Udaya's (or $\mathcal{S}_e(1)$) and the Kasami (large set) sequences which provide the same maximum correlation as $\mathcal{Z}_e(3)$. If $D \geq 4$, on the other hand, $\mathcal{Z}_e(D)$ gets far from the asymptotic bounds. However, it still provides much smaller normalized maximum correlation than those of $\mathcal{S}_e(\rho)$ with $\rho \geq 3$ whose maximum correlation increases exponentially.

In every region of $\log_L M$, we conclude that $\mathcal{Z}_e(D)$ is the best known binary sequence family for even n in terms of maximum correlation and family size. In particular, the Kerdock sequences give maximum family size for

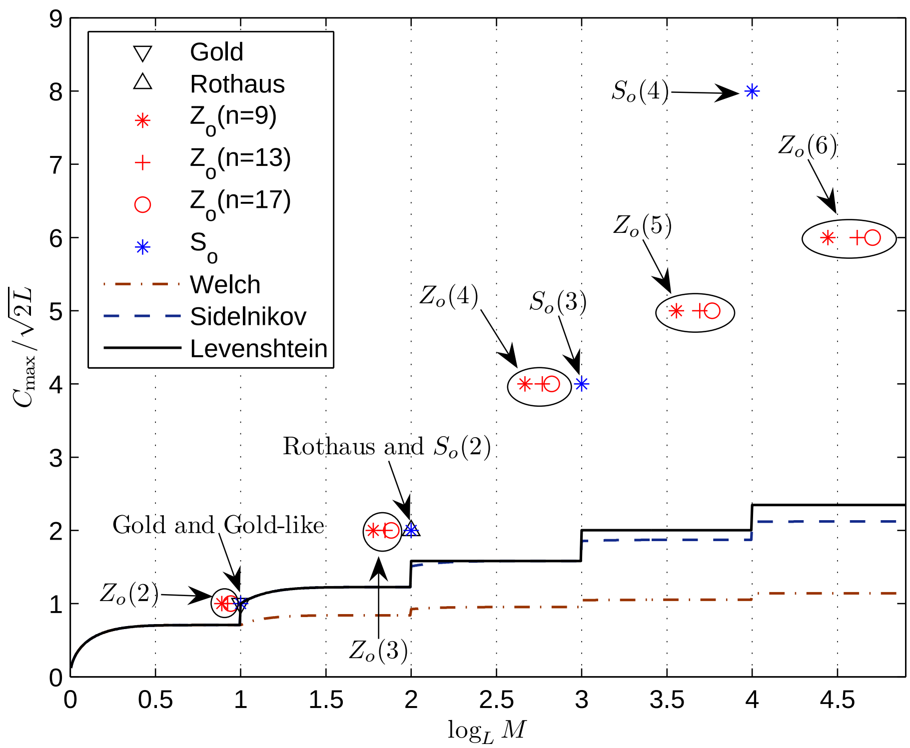

Fig. 5. Asymptotic normalized maximum correlation and the bounds (n is odd)

sufficiently large n asymptotically achieving all the three bounds.

6. CONCLUSION

The asymptotic Welch, Sidelnikov, and Levenshtein bounds have been studied for a binary sequence family of size $M = L^l$ with a positive real value l. The bounds have been compared with the asymptotic parameters of several known binary sequence families and the asymptotic optimality of the families have been discussed from the comparison. In conclusion, it is shown that the Gold-like (or $\mathcal{S}_o(\rho)$) and the Kerdock sequences are the best of all known binary sequence families for odd and even n, respectively, achieving the asymptotic Levenshtein bound. Furthermore, it is also shown that $\mathcal{S}_o(\rho)$ of $\rho \leq 3$ and $Z_e(D)$ are the best known families in terms of both maximum correlation and family size.

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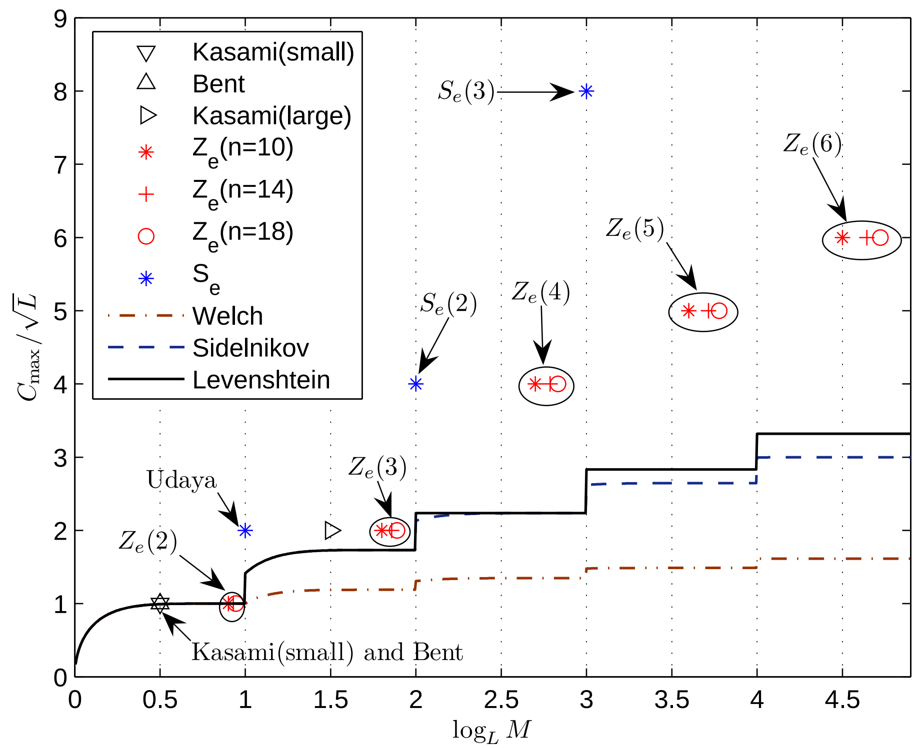

Fig. 6. Asymptotic normalized maximum correlation and the bounds (n is even)

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