NON-HYPERELLIPTIC MODULAR CURVES OF GENUS 3

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Abstract. A curve C defined over $\mathbb{Q}$ is modular of level N if there exists a non-constant morphism $X_1(N) \longrightarrow C$ defined over $\mathbb{Q}$ for some positive integer N. We present an algorithm to compute explicitly equations for modular non-hyperelliptic curves defined over $\mathbb{Q}$ of genus 3. Let C be a modular curve of level N, we say that C is new if the corresponding morphism between $J_1(N)$ and $\mathrm{Jac}(C)$ factorizes through the new part of $J_1(N)$. We compute equations of 44 non-hyperelliptic new modular curves of genus 3, that we conjecture to be the complete list of this kind of curves. Furthermore, we describe some aspects of non-new modular curves.

1. Introduction

Let N be a positive integer and $X_1(N)$ (resp. $X_0(N)$) be the classical modular curve corresponding to the modular group $\Gamma_1(N)$ (resp. $\Gamma_0(N)$). Many papers have been already devoted to the problem of finding $\mathbb{Q}$-rational models for these modular curves and their quotients [Shimura 95, González 91, Murabayashi 92, Hasegawa 97, Hasegawa and Hashimoto 96, Furumoto and Hasegawa 99]. In this work we are interested in modular curves defined over $\mathbb{Q}$ which are dominated over $\mathbb{Q}$ by $X_1(N)$.

In [González-Jiménez and González 03, Baker et al. 05] the concept of new modular curves is introduced. That is, curves dominated by $X_1(N)$ such that the corresponding morphism on their jacobians factorizes through the new part of the jacobian of $X_1(N)$. For the case of genus 1, the concept of new modular curve and modular curve is equivalent. In a series of papers, Wiles et al. [Wiles 95, Taylor and Wiles 95, Breuil et al. 01] proved that every elliptic curve defined over $\mathbb{Q}$ is modular. Then there are infinitely many new modular curves over $\mathbb{Q}$ of genus 1. In contrast to new modular elliptic curves, for a fixed genus $g \geq 2$, the set of new modular curves of genus g (up to $\mathbb{Q}$-isomorphism) is finite and computable [Baker et al. 05], and in the case of genus 2 [Baker et al. 05, González-Jiménez and González 03] provide a complete list of new modular curves. On the one hand, using the procedure described in [Baker et al. 05], it is theoretically possible to compute all the new modular curves of a given genus $g \geq 3$, but the enormous number of possibilities make practical computations impossible. On the other hand, the methods described in [Baker et al. 05] do not give any information about the highest level N_g that can appear for possible modular curves of genus g. In [Baker et al. 05] the authors study in detail new modular hyperelliptic curves and they construct what might be the complete list of all new modular hyperelliptic curves of all genera. In this paper, we study the simplest case

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of non-hyperelliptic new modular curves, i.e. the case of genus 3 (smooth plane quartics). We therefore restrict our attention to the computation of all non-hyperelliptic new modular curves of genus 3 up to a fixed level (see appendix).

This paper is organized as follows: In Sections 2 and 3, we review the necessary technical background about modular curves and non-hyperelliptic genus 3 curves respectively. In Section 3, we present a method that allow us to recognize if an abelian 3-fold A (with some extra conditions) corresponds to a non-hyperelliptic curve of genus 3. In Section 4, we apply this method to compute all the new modular non-hyperelliptic curves of genus 3 up to certain level bounded by constants depending on the splitting behaviour of their jacobian. In Section 5, we present some examples that show the difficulty of the non-new modular case. We conclude this paper with an appendix that gives equations of 44 non-hyperelliptic new modular curves of genus 3, and we conjecture to be the complete list of this kind of curves.

Remark 1. All curves and varieties in this paper are smooth and projective unless otherwise specified. If X is a variety over a field k, let $\Omega^1 = \Omega^1_{X/k}$ denote the sheaf of regular 1-forms. $G_{\mathbb{Q}}$ will denote $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Along the paper we will use the labelling of modular abelian varieties as it was introduced in [Baker et al. 05].

2. Modular curves

Let $N > 2$ be an integer and

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) |\ c \equiv 0 \mod N \right\},$$

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) |\ a \equiv d \equiv 1 \mod N \right\},$$

the classical congruence subgroups of level N. Let $X_0(N)$ and $X_1(N)$ denote models over $\mathbb{Q}$ of the associated modular curves. Therefore, since $\Gamma_1(N)$ is a subgroup of $\Gamma_0(N)$ there exists a non-constant $\mathbb{Q}$-morphism from $X_1(N)$ onto $X_0(N)$.

Let $S_2(N)$ be the set of cusp forms of weight 2 for the congruence subgroup $\Gamma_1(N)$. The map

$$\omega : S_2(N) \longrightarrow H^0(X_1(N), \Omega^1),\ f(q) \longmapsto f(q)\, dq/q$$

induces an isomorphism between the vector spaces $S_2(N)$ and $H^0(X_1(N), \Omega^1)$.

Using diamond operators and character theory, the space $S_2(N)$ of cusp forms admits the following decomposition

$$S_2(N) = \bigoplus_{\varepsilon} S_2(N, \varepsilon),$$

where ε runs over all Dirichlet characters modulo N and $S_2(N, \varepsilon)$ denotes the complex vector space of cusp forms of weight 2 on $\Gamma_1(N)$ with nebentypus ε. With this notation, $H^0(X_0(N), \Omega^1)$ is canonically isomorphic to $S_2(N, 1)$.

If $M|N$ and $d|\frac{N}{M}$, then $z \mapsto d \cdot z$ induces a morphism $X_1(N) \longrightarrow X_1(M)$, which also induces morphisms $S_2(M) \longrightarrow S_2(N)$ and $J_1(M) \longrightarrow J_1(N)$, where $J_1(N) := \mathrm{Jac}(X_1(N))$. The old subspace $S_2^{\mathrm{old}}(N)$ of $S_2(N)$ is defined as the sum of the images of all such maps $S_2(M) \longrightarrow S_2(N)$ for all d and M such that $M|N, M \neq N$ and

$d|\frac{N}{M}$. Similarly we define the old subvariety $J_1^{\mathrm{old}}(N)$ of $J_1(N)$. Let $S_2^{\mathrm{new}}(N)$ be the orthogonal complement to $S_2^{\mathrm{old}}(N)$ with respect to the Petersson inner product in $S_2(N)$. For $n \geq 1$ with $\gcd(N,n) = 1$, there exist correspondences T_n on $X_1(N)$, which induce endomorphisms of $S_2(N)$ and of $J_1(N)$ known as Hecke operators, also denoted by T_n. There exists a unique basis of $S_2^{\mathrm{new}}(N)$ consisting of eigenforms with respect to all the T_p (for $\gcd(N,p) = 1$), i.e. cusp forms $f(q) = q + \sum_{i\geq 2} a_i q^i$ such that $T_n(f) = a_n f$ whenever $\gcd(n,N) = 1$. The elements of this basis are called newforms of level N, and this basis will be denoted by New_N. To the newform $f(q) = q + \sum_{i\geq 2} a_i q^i$, let $K_f = \mathbb{Q}(\{a_n\})$ be the algebraic number field generated by the coefficients a_n of f. Then K_f is totally real if the nebentypus of f is trivial and a CM field if it is non-trivial. Let $I_f = \{\sigma_1, \ldots, \sigma_d\}$ be the set of all isomorphisms of K_f into $\mathbb{C}$, and $\{{}^{\sigma_1}f, \ldots, {}^{\sigma_d}f\}$ be the complete set of newforms conjugate to f over $\mathbb{Q}$. Shimura [Shimura 73] attached to $f \in \mathrm{New}_N$ an abelian variety A_f defined over $\mathbb{Q}$ with the following properties: A_f is a $\mathbb{Q}$-simple factor of $J_1(N)$, $\dim(A_f) = d$ and $H^0(A_f, \Omega^1) \simeq \sum_{\sigma\in I_f} \mathbb{C}\,\omega({}^{\sigma}f)$. If the nebentypus of f is non-trivial, then K_f is a CM field and thus the dimension of A_f is even.

The definition of A_f directly implies the existence of a surjective $\mathbb{Q}$-morphism

$$\pi_f : J_1(N) \twoheadrightarrow A_f\,.$$

Furthermore, the decomposition of $J_1(N)$ over $\mathbb{Q}$ is:

$$J_1(N) \stackrel{\mathbb{Q}}{\sim} \prod_{M|N} \prod_{f\in \mathrm{New}_M/G_{\mathbb{Q}}} A_f^{\sigma_0(\frac{N}{M})},$$

where $\sigma_0(n)$ denotes the number of positive divisors of n. We will denote by

$$J_1^{\mathrm{new}}(N) \stackrel{\mathbb{Q}}{\sim} \prod_{f\in \mathrm{New}_N/G_{\mathbb{Q}}} A_f\,.$$

Definition 1. [González-Jiménez and González 03, Baker et al. 05] An abelian variety A over $\mathbb{Q}$ is said to be *modular of level* N if there exists a surjective $\mathbb{Q}$-morphism

$$\nu : J_1(N) \twoheadrightarrow A\,.$$

In that case, we say that A is *new (of level* N*)*, if ν factorizes through $J_1^{\mathrm{new}}(N)$ over $\mathbb{Q}$. Then we have

$$S_2(A) := \nu^* H^0(A, \Omega^1)\,\frac{q}{dq} \subseteq S_2(N)^{\mathrm{new}}\,.$$

Definition 2. [González-Jiménez and González 03, Baker et al. 05] A non-singular curve C defined over $\mathbb{Q}$ is said to be *modular of level* N, if there exists a non-constant $\mathbb{Q}$-morphism

$$\pi : X_1(N) \twoheadrightarrow C\ .$$

The modular curve C is then said to be *new of level* N if its jacobian $\mathrm{Jac}(C)$ is new of level N.

Similarly, for a modular curve and their associated dominant morphism π we define

$$S_2(C) := \pi^* H^0(C, \Omega^1)\,\frac{q}{dq} \subseteq S_2(N)\,.$$

For the modular curve C, the following diagram commutes:

$$\begin{array}{ccc} J_1(N) & \xrightarrow{\pi_*} & \mathrm{Jac}(C) \\ \uparrow & & \uparrow \\ X_1(N) & \xrightarrow{\pi} & C \end{array}$$

The converse of the above sentences is not true in general (cf. [González-Jiménez and González 03, section 7]).

As a first step to understand the structure of new modular curves, the authors of [González-Jiménez and González 03] showed that the set of new modular curves of genus 2 over $\mathbb{Q}$ is finite, and that there are exactly 149 such curves whose jacobians are $\mathbb{Q}$-simple. In [Baker et al. 05] the case of genus 2 is completed, showing that there are exactly 213 new modular curves of genus 2. Furthermore, [Baker et al. 05] generalized the above approach for new modular curves with fixed genus $g \geq 2$ obtaining the following result:

Theorem 1. [Baker et al. 05] *For each integer $g \geq 2$, the set of new modular curves over $\mathbb{Q}$ of genus g is finite and computable.*

3. Non-hyperelliptic curves of genus 3

Since parts of the theory concerning non-hyperelliptic genus 3 curves is not easily available in modern publications, we include here explicitly almost all the theory we use along the article. Specifically, we present some facts about holomorphic differentials, canonical embeddings and automorphism groups.

3.1. **Holomorphic differentials.** The goal of this section is to deal with holomorphic differentials of non-hyperelliptic curves of genus 3 in order to compute the equation of the image of the canonical embedding. Proposition 1 is the main result presented in this section. It will allow from a given curve X to decide if an abelian 3-fold A defined over a field k, which is dominated over k by $\mathrm{Jac}(X)$ and given by its space of holomorphic differentials $H^0(A,\Omega^1)$, is k-isogenous to the jacobian of a genus 3 non-hyperelliptic curve C.

The following classical result on plane affine models of algebraic curves (defined over a field k of characteristic 0) can be used to compute a basis of $H^0(C,\Omega^1)$:

Lemma 1. [Brieskorn and Knörrer 86, p.630] *Let C be an irreducible plane algebraic curve of degree n with an affine equation $f(x,y) = 0$, and let the coordinates be chosen so that the partial derivative $f_y(x,y)$ of $f(x,y)$ with respect to y does not vanish identically. Let C' be the Riemann surface which results from C by resolution of singularities. Then the nonvanishing differentials of 1st kind of C' are just the differential forms*

$$\frac{h(x,y)}{f_y(x,y)}\,dx\,,$$

where $h(x,y) = 0$ is the equation of a curve of degree $n-3$ adjoint to C.

In the sequel, let k be a field of characteristic 0, C be a non-hyperelliptic curve of genus 3 defined over k and let $\{\omega_1, \omega_2, \omega_2\}$ be a basis of the space $H^0(C, \Omega^1)$ of holomorphic differential forms on C. The canonical embedding of C with respect to this basis is given by

$$\begin{array}{rccl} \phi: & C & \longrightarrow & \mathbb{P}^2 \\ & P & \longmapsto & \phi(P) := (\omega_1(P) : \omega_2(P) : \omega_3(P)), \end{array}$$

where $\omega(P) = g(P)$ for any expression $\omega = g dt_P$, with $g, t_P \in k(C)$ and t_P a local parameter at P. The image $\phi(C)$ of C by the canonical embedding is a smooth plane quartic, and conversely any smooth plane quartic is the image by the canonical embedding of a genus 3 non-hyperelliptic curve.

From now on, let C be a smooth plane quartic defined over k by the homogeneous equation $F(X, Y, Z) = 0$. The regular differentials of C correspond to intersection divisors of the curve C and the lines of $\mathbb{P}^2$, i.e. the positive divisor (of degree 4) whose support contains exactly the four points of intersection between a line l and the curve C. Up to scalar multiplication there is a canonical basis of $H^0(C, \Omega^1)$ given by the three regular differentials whose divisors are respectively the intersection divisors between the curve C and the lines X, Y, Z, i.e.

$$(C \cdot V(X)), \ (C \cdot V(Y)), \ (C \cdot V(Z)),$$

where $V(G)$ denotes the algebraic variety defined by the equation $G(X, Y, Z) = 0$. In affine coordinates x, y, these are the differentials

$$\frac{x}{f_y(x,y)}\, dx\,, \frac{y}{f_y(x,y)}\, dx\,, \frac{1}{f_y(x,y)}\, dx\,,$$

where $f(x, y) = F(x, y, 1) = 0$ is an affine model of C and $f_y(x, y)$ the partial derivative of $f(x, y)$ with respect to y.

Using the homogenization $F(X, Y, Z) = 0$ of C, i.e. $x := \frac{X}{Z}$, $y := \frac{Y}{Z}$ where $Z = 0$ is the line at infinity, the holomorphic differentials $\frac{h(x,y)}{f_y(x,y)}\, dx$ from Lemma 1 have the homogeneous form:

$$\frac{H(X, Y, Z)(Z\, dX - X\, dZ)}{F_Y(X, Y, Z)},$$

where $H(X, Y, Z) = 0$ is the homogeneous equation of $h(x, y) = 0$.

We then have

Lemma 2. *For a smooth plane quartic $C : F(X, Y, Z) = 0$ we have*

$$H^0(C, \Omega^1) \ = \ \langle X\Psi_F(X, Y, Z), Y\Psi_F(X, Y, Z), Z\Psi_F(X, Y, Z)\rangle\,,$$

where $\Psi_F(X, Y, Z) = \dfrac{Z\, dX - X\, dZ}{F_Y(X, Y, Z)}$.

We are interested in deciding when an abelian 3-fold is isogenous to the jacobian of a non-hyperelliptic genus 3 curve. The following proposition gives us an answer.

Proposition 1. *Let X be a curve and A an abelian 3-fold both defined over k such that there exists $\nu : \mathrm{Jac}(X) \twoheadrightarrow A$ defined over k. The following are equivalent:*

(i) *There exist a non-hyperelliptic genus 3 curve C defined over k and a non-trivial k-morphism $\pi : X \twoheadrightarrow C$ such that $\mathrm{Jac}(C) \overset{k}{\sim} A$.*

(ii) *For every basis* $\{f_1(z), f_2(z), f_3(z)\}\, dz$ *of* $\nu^* H^0(A, \Omega^1)$ *there exist a* k*-rational smooth plane quartic* $F(X, Y, Z) = 0$ *and a constant* $c_F \in k^*$ *such that* $F(f_1, f_2, f_3) = 0$ *and*

$$\psi_F(z) := \frac{f_3(z) f_1'(z) - f_1(z) f_3'(z)}{F_Y(f_1(z), f_2(z), f_3(z))} = c_F\,,$$

where $f_j'(z)$ *denotes the derivative of* $f_j(z)$ *with respect to* z*.*

Proof. Let first prove that (i) implies (ii). Let $\{f_1(z), f_2(z), f_3(z)\}\, dz$ be a basis of $\nu^* H^0(A, \Omega^1)$. From $\mathrm{Jac}(C) \stackrel{k}{\sim} A$ it follows $\pi^* H^0(C, \Omega^1) = \nu^* H^0(A, \Omega^1)$. Since the curve C is non-hyperelliptic there exists an irreducible and non-singular homogeneous polynomial $F(X, Y, Z) \in k[X, Y, Z]$ of degree 4 such that the image of the canonical embedding of C (with respect to the basis $\{f_1(z), f_2(z), f_3(z)\}\, dz$) is defined by $F(X, Y, Z) = 0$ and thus is $F(f_1, f_2, f_3) = 0$. Now, using Lemma 2 we have

$$\Psi_F(f_1(z), f_2(z), f_3(z)) = \frac{f_3(z) f_1'(z) - f_1(z) f_3'(z)}{F_Y(f_1(z), f_2(z), f_3(z))}\, dz =: \psi_F(z)\, dz\,.$$

Therefore

$$\begin{array}{rcl} \nu^* H^0(A, \Omega^1) & = & \langle f_1(z), f_2(z), f_3(z)\rangle\, dz \\ & = & \pi^* H^0(C, \Omega^1) \quad = \quad \langle f_1(z)\psi_F(z), f_2(z)\psi_F(z), f_3(z)\psi_F(z)\rangle\, dz\,, \end{array}$$

and hence there exists a matrix $M \in \mathrm{GL}_3(k)$ with

$$M \begin{pmatrix} f_1(z)\, dz \\ f_2(z)\, dz \\ f_3(z)\, dz \end{pmatrix} = \begin{pmatrix} f_1(z)\psi_F(z)\, dz \\ f_2(z)\psi_F(z)\, dz \\ f_3(z)\psi_F(z)\, dz \end{pmatrix}.$$

The only possibility for M to solve this equation is in fact $M = \psi_F(z)\, I_3$ where I_3 is the identity matrix of $\mathrm{GL}_3(k)$. Therefore $\psi_F(z) = c_F \in k^*$. Then we have proved that (i) implies (ii).

Now we have to prove that (ii) implies (i). Let g be the genus of X. Since X and A are both defined over k there exists a basis $\{f_1(z), \ldots, f_g(z)\}\, dz$ of $H^0(X, \Omega^1)$ defined over k such that $\{f_1(z), f_2(z), f_3(z)\}\, dz$ is a basis of $\nu^* H^0(A, \Omega^1)$ defined over k. Then there exists a smooth plane quartic $C : F(X, Y, Z) = 0$ defined over k such that $\pi^* H^0(C, \Omega^1) = \langle f_1(z), f_2(z), f_3(z)\rangle \psi_F(z) dz = \langle f_1(z), f_2(z), f_3(z)\rangle dz = \nu^* H^0(A, \Omega^1)$ since $\psi_F(z) = c_F \in k^*$. The following map

$$\begin{array}{ccccc} X & \to & \mathbb{P}^{g-1} & \to & \mathbb{P}^2 \\ z & \to & [f_1(z) : \cdots : f_g(z)] & \to & [f_1(z) : f_2(z) : f_3(z)] \end{array}$$

defines a non-constant k-morphism from X to C such that $\mathrm{Jac}(C) \stackrel{k}{\sim} A$. Therefore we have proved that (ii) implies (i). □

If there exists an abelian variety A with $H^0(A, \Omega^1) = \langle f_1(z), f_2(z), f_3(z)\rangle\, dz$ and a smooth plane quartic $C : F(X, Y, Z) = 0$ with $F(f_1(z), f_2(z), f_3(z)) = 0$ and $\psi_F(z) \notin \mathbb{C}$, then the jacobian of the curve C must not be necessary isogenous to the abelian variety A. For example: if we start with a $\mathbb{Q}$-basis of $H^0(A, \Omega^1) = \langle f_1(z), f_2(z), f_3(z)\rangle\, dz$ where A is the modular abelian 3-fold $A_{120A_{\{0,1,0\}}} \times E_{120A}$ we obtain a smooth plane quartic $C : F(X, Y, Z) = 0$ defined over $\mathbb{Q}$ with $F(f_1(z), f_2(z), f_3(z)) = 0$ but $\psi_F(z) \notin \mathbb{C}$. In fact C is modular and $\mathrm{Jac}(C) \not\sim A$, but $\mathrm{Jac}(C) \stackrel{\mathbb{Q}}{\sim} A_{30A_{\{0,2\}}} \times X_0(15)$, i.e. C is non-new (see Example 5).

3.2. **Automorphism group.** In the following, let C be a smooth, projective curve of genus $g \geq 2$, defined over an algebraic closed field K of characteristic 0. Hurwitz [Hurwitz 92] proved that the full automorphism group of C is a finite group of order bounded by $84(g-1)$.

The Hurwitz formula provides the following result about the largest order of cyclic subgroups of $\mathrm{Aut}(C)$ depending on the genus of the quotient curve:

Proposition 2. [Hurwitz 88, Kuribayashi and Komiya 79] *Let H be a cyclic subgroup of $\mathrm{Aut}(C)$ and denote by $\tilde{g}$ the genus of C/H, $m = |H|$ and r be the number of ramification points of the Galois cover $C \longrightarrow C/H$. The following holds:*

(i) *If $\tilde{g} \geq 2$, then $m \leq g-1$,*
(ii) *If $\tilde{g} = 1$, then $m \leq 2(g-1)$,*
(iii) *If $\tilde{g} = 0$ and* $\begin{cases} r \geq 5, & \text{then } m \leq 2(g-1)\,, \\ r = 4, & \text{then } m \leq 6(g-1)\,, \\ r = 3, & \text{then } m \leq 10(g-1)\,. \end{cases}$

The last bound $m \leq 10(g-1)$ was improved by Wiman [Wiman 95] in 1895 to $m \leq 2(2g+1)$. Furthermore, in the case of prime order p, then $p \leq g$ or $p = g+1$ or $p = 2g+1$ (cf. [Farkas and Kra 92, Proposition V.2.14]).

Another nice application of the Hurwitz formula is the following:

Proposition 3. [Accola 70] *Let H and H_j $(1 \leq j \leq k)$ be subgroups of $\mathrm{Aut}(C)$ such that $H = \bigcup_{j=1}^{k} H_j$ and $H_i \cap H_l = \{id\}$ if $i \neq l$. Denote by $m_j := |H_j|$, $m := |H|$, $\tilde{g}$ the genus of C/H and $\tilde{g}_j$ the genus of C/H_j. Then,*

$$(k-1)g + m\tilde{g} = \sum_{j=1}^{k} m_j \tilde{g}_j$$

From now on let C be a non-hyperelliptic curve of genus 3. Then,

Proposition 4. [Kuribayashi and Komiya 79] *Any involution σ of a non-hyperelliptic curve C of genus 3 is a bielliptic involution (i.e. the genus of $C/\langle\sigma\rangle$ is 1).*

An application of the above results is the following: Let $H := \{id, \sigma_1\} \cup \{id, \sigma_2\} \cup \{id, \sigma_3\} \simeq (\mathbb{Z}/2\mathbb{Z})^2$ where the σ_i are distinct involutions of some non-hyperelliptic genus 3 curve C. Using Proposition 3 and 4 with $k = 3, \tilde{g}_i = 1, \tilde{m_i} = 2, g = 3$ and $m = 4$ we obtain

$$6 + 4\tilde{g} = 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1\,,$$

hence $\tilde{g} = 0$.

Now, suppose C has a non-trivial automorphism σ. The genus of $C/\langle\sigma\rangle$ is either 0, 1 or 2 (Hurwitz's formula). Using Proposition 2, we conclude that if the genus of $C/\langle\sigma\rangle$ is 2 then $\sigma^2 = id$, thus by Proposition 4, C is hyperelliptic, in contradiction with the hypothesis. Therefore $C/\langle\sigma\rangle$ has genus 1 or 0, i.e. C has a Galois cyclic cover to a projective line or to an elliptic curve. Using Proposition 2, if $\mathrm{Aut}(C)$ has an element of order > 4, then $C/\langle\sigma\rangle$ has genus 0.

In fact, for a non-hyperelliptic curve C of genus 3 we could determine the possible order of any automorphism of C.

Proposition 5. [Kuribayashi and Komiya 79] *Let C be a non-hyperelliptic curve of genus* 3 *and* $\sigma \in \mathrm{Aut}(C)$. *Then* $\mathrm{ord}\,(\sigma) \in \{1,2,3,4,6,7,8,9,12\}$.

In the case of an abelian subgroup of the automorphism group such that the quotient curve is an elliptic curve we could say something more:

Proposition 6. *Let C be a non-hyperelliptic curve of genus* 3 *and G an abelian subgroup of* $\mathrm{Aut}(C)$ *such that the genus of C/G is* 1 . *Then G is cyclic of order* 2, 3 *or* 4.

Proof. According to the full automorphism group [Kuribayashi and Komiya 79] that can occur for $\mathrm{Aut}(C)$, we have $|\mathrm{Aut}(C)| \in \{1,2,3,4,6,7,8,9,16,24,48,96,168\}$. Then we can assume that there exist non-negative integers $a \leq 5, b \leq 2$ and $c \leq 1$ such that $|G| = 2^a \cdot 3^b \cdot 7^c$. Using Proposition 2, the integer c must be equal to 0, otherwise there would exist a cyclic subgroup $\langle\sigma\rangle$ of order 7 and hence $g(C/G) = 0$. So, the only possible orders of G are

$$2, 3, 2^2, 2 \cdot 3, 2^3, 3^2, 2^2 \cdot 3, 2^3 \cdot 3, 2^4, 2^5, 2^4 \cdot 3, 2^5 \cdot 3\,.$$

G cannot be equal to $(\mathbb{Z}/2\mathbb{Z})^2$ because $g(C/(\mathbb{Z}/2\mathbb{Z})^2) = 0$. Furthermore, G cannot be of order $2^3, 2^4, 2^5$ since any such abelian group has $(\mathbb{Z}/2\mathbb{Z})^2$ or a cyclic subgroup of order > 4 as subgroups. The only possibility for 3^2 to divide $|G|$ is that $G = \mathbb{Z}/9\mathbb{Z}$ (see full automorphism group [Kuribayashi and Komiya 79]). Also the above case cannot happens because $g(C/(\mathbb{Z}/9\mathbb{Z})) = 0$. The other orders $2 \cdot 3, 2^2 \cdot 3, 2^3 \cdot 3, 2^4 \cdot 3, 2^5 \cdot 3$ cannot occur, since the corresponding groups have at least a cyclic subgroup of order > 4. □

4. Non-hyperelliptic modular curves of genus 3

4.1. **Modular Criteria.** For non-hyperelliptic curves of genus 3, Proposition 7 and Lemma 3 will provide us with some effective criteria to determine when a factor of $J_1(N)$ defined over $\mathbb{Q}$ is $\mathbb{Q}$-isogenous to the jacobian of a non-hyperelliptic modular curve of genus 3.

Proposition 7. *Let A be a* 3*-dimensional quotient of* $J_1(N)$. *The following are equivalent:*

(i) *There exist a non-hyperelliptic genus* 3 *curve C and a non-trivial $\mathbb{Q}$-morphism* $\pi : X_1(N) \twoheadrightarrow C$ *such that* $\mathrm{Jac}(C) \stackrel{\mathbb{Q}}{\sim} A$.

(ii) *There exist a basis* $\{f_1, f_2, f_3\}$ *for* $S_2(A)$ *and a smooth plane quartic* $C : F(X,Y,Z) = 0$ *such that* $F(f_1(q), f_2(q), f_3(q)) = 0$ *and*

$$\psi_F(q) := q\,\frac{f_3(q)f_1'(q) - f_1(q)f_3'(q)}{F_Y(f_1(q), f_2(q), f_3(q))} \in \mathbb{Q}^*\,.$$

Proof. It is a straightforward consequence of Proposition 1 just replacing X by $X_1(N)$ and taking into account that if we make the change of variable $q = e^{2\pi i z}$ then

$$\psi_F(f_1(q), f_2(q), f_3(q)) = \psi_F(q).$$

□

It can happen that for $S_2(A) = \langle f_1, f_2, f_3\rangle_{\mathbb{Q}} \subseteq S_2(N)$ there exists a (plane) non-hyperelliptic curves of genus 3 with equation $C_d : F_d(X,Y,Z) = 0$ of degree $d \geq 5$ such that $F_d(f_1(q), f_2(q), f_3(q)) = 0$. In that case, the image $C' : F'(X,Y,Z) = 0$ of the canonical embedding of C is a modular curve (since the inclusion between the function fields $\mathbb{Q}(C') \subseteq \mathbb{Q}(X_1(N))$ implies the existence of a $\mathbb{Q}$-morphism from $X_1(N)$ onto C'), however $\mathrm{Jac}(C')$ is in general not necessary $\overline{\mathbb{Q}}$-isogenous to A (see Examples 3 and 4).

Lemma 3. *Let C be a non-hyperelliptic new modular curve of genus 3 and level N. Let $\pi : X_1(N) \twoheadrightarrow C$ the corresponding modular parametrization of C.*

(i) *Then there exist $h_1, h_2, h_3 \in S_2(N)$ with rational q-expansion*

$$\begin{cases} h_1(q) = q + O(q^2) \\ h_2(q) = q^2 + O(q^3) \\ h_3(q) = O(q^3)\,, \end{cases}$$

such that $\pi^* H^0(C, \Omega^1) = \langle h_1(q), h_2(q), h_3(q)\rangle \frac{dq}{q}$ with $\mathrm{ord}_q h_3 \leq 5$. Furthermore, if $\mathrm{Jac}(C)$ is $\mathbb{Q}$-simple, then $\mathrm{ord}_q h_3 < 5$.

(ii) *With respect to the canonical embedding*

$$\phi : q \longmapsto (h_1(q) : h_2(q) : h_3(q))\,,$$

$\phi(C)$ is a smooth plane quartic given by an equation $C : F(X,Y,Z) = 0$ with the $\mathbb{Q}$-rational point $P_\infty = (1:0:0) \in \phi(C)(\mathbb{Q})$. The point P_∞ is a flex (resp. a hyperflex) if $\mathrm{ord}_q h_3 \geq 4$ (resp. $\mathrm{ord}_q h_3 = 5$). Furthermore, by approriate normalization of the equation of the curve, we can assume $\psi_F(q) = 1$.

Remark 2. If $\mathrm{Jac}(C)$ is $\mathbb{Q}$-simple then P_∞ could not be a hyperflex of C, while P_∞ could be a hyperflex of C if A is not $\mathbb{Q}$-simple (c.f. $C^{A}_{39A_{\{0,6\}}}$, $C^{A,C,D}_{99}$ from Table 1 in the appendix). For all the different $\mathbb{Q}$-splitting behaviour of jacobian varieties of non-hyperelliptic modular curves of genus 3 there are curves for which P_∞ is an ordinary flex.

Proof. (i) We are going to split the proof in three cases attending to the decomposition of $\mathrm{Jac}(C)$ over $\mathbb{Q}$. These are the possible cases:

Case **A**. $\mathrm{Jac}(C)$ is $\mathbb{Q}$-simple. Then $\mathrm{Jac}(C) \overset{\mathbb{Q}}{\sim} A_f$ with $f \in S_2(N,1)$.

Case **AE**. $\mathrm{Jac}(C) \overset{\mathbb{Q}}{\sim} E \times A$, where E is an elliptic curve over $\mathbb{Q}$ and A is a $\mathbb{Q}$-simple 2-fold. Then there exists $g \in S_2(N,1)$ such that $A_g \overset{\mathbb{Q}}{\sim} E$ and $f \in S_2(N,\varepsilon)$ such that $A_f \overset{\mathbb{Q}}{\sim} A$ and $\mathrm{ord}\,(\varepsilon) \in \{1,2,3,4,6\}$ (since $\mathbb{Q}(\varepsilon) \subset K_f$).

Case **EEE**. $\mathrm{Jac}(C) \overset{\mathbb{Q}}{\sim} E_1 \times E_2 \times E_3$, where E_1, E_2, E_3 are elliptic curves defined over $\mathbb{Q}$. Then there exists $f_i \in S_2(N,1)$ such that $A_{f_i} \overset{\mathbb{Q}}{\sim} E_i$, $i = 1,2,3$.

Following [Baker et al. 05, Corollary 7.3 (i)], there exists a basis $\{g_1, g_2, g_3\}$ of $\pi^* H^0(C, \Omega^1)$ such that $g_i(q) = q + \sum_{n\geq 2} a_n^{(i)} q^n$ and since C is non-hyperelliptic,

(1) *it is not possible that* $a_2^{(1)} = a_2^{(2)} = a_2^{(3)}$.

We are going to apply (1) to our three cases:

Case **A**. Let $f(q)=\sum_{n\geq 1} a_n q^n$. Then by (1) we have $a_2\notin\mathbb{Q}$, that is, $K_f=\mathbb{Q}(a_2)$. In this case $S_2(A_f)=\langle f, {}^{\sigma}f, {}^{\beta}f\rangle$ where $\mathrm{Aut}(K_f)=\{id,\sigma,\beta\}$. Now we construct an explicit $\mathbb{Q}$-basis $\{h_1,h_2,h_3\}$ for $S_2(A_f)$. Let $p(x)=x^3+ax^2+bx+c$ be the minimal polynomial of a_2 and

$$g_i=\frac{1}{3}\,\mathrm{Tr}_{K_f/\mathbb{Q}}(a_2^{i-1}\,f)\,,\qquad i=1,2,3.$$

Then g_1,g_2,g_3 have rational q-expansion and $\pi^*\,H^0(C,\Omega^1)=\langle g_1(q),g_2(q),g_3(q)\rangle\,\frac{dq}{q}$.

Now, let

$$\begin{aligned}
&h_1(q)=g_1(q)\,,\\
&h_2(q)=\frac{18}{\mathrm{disc}(p'(x))}(g_2(q)+\frac{a}{3}g_1(q))\,,\\
&h_3(q)=g_3(q)-\frac{1}{3}(a^2-2b)g_1(q)+\frac{1}{9}(2a^3-7ab+9c)g_2(q)=A\gamma_3q^3+Aq^4+O(q^5)\,,
\end{aligned}$$

where $a_3=\alpha_3+\beta_3\,a_2+\gamma_3\,a_2^2$, since $a_3\in K_f=\mathbb{Q}(a_2)$, and

$$A=\frac{2}{3}\frac{\mathrm{disc}(p(x))}{\mathrm{disc}(p'(x))}.$$

Since $p(x)$ has three different real roots, $\mathrm{disc}(p(x)),\mathrm{disc}(p'(x))\neq 0$, then $A\neq 0$. After normalizing, we have $h_3(q)=q^3+O(q^4)$ in the case $\gamma_3\neq 0$ and $h_3(q)=q^4+O(q^5)$ in the case $\gamma_3=0$.

Case **AE**. Let be $K_f=\mathbb{Q}(\sqrt{d})$, $\mathrm{Aut}(K_f)=\{id,\sigma\}$ and

$$\begin{aligned}
f(q)&=q+(A_2+B_2\sqrt{d})q^2+O(q^3)\,,\\
{}^{\sigma}f(q)&=q+(A_2-B_2\sqrt{d})q^2+O(q^3)\,,\\
g(q)&=q+c_2q^2+O(q^3)\,.
\end{aligned}$$

Then by (1) there exist two possibilities depending if $B_2=0$ or $B_2\neq 0$.

- If $B_2\neq 0$, then we choose

$$\begin{aligned}
h_1(q)&=\tfrac{f(q)+{}^{\sigma}f(q)}{2}=q+O(q^2)\,,\\
h_2(q)&=\tfrac{f(q)-{}^{\sigma}f(q)}{2B_2\sqrt{d}}=q^2+O(q^3)\,,\\
h_3(q)&=g(q)-h_1(q)-(c_2-A_2)h_2(q)=O(q^3)\,.
\end{aligned}$$

- If $B_2=0$, by (1) we have $A_2\neq c_2$, hence

$$\begin{aligned}
h_1(q)&=\tfrac{f(q)+{}^{\sigma}f(q)}{2}=q+O(q^2)\,,\\
h_2(q)&=\tfrac{g(q)-h_1(q)}{c_2-A_2}=q^2+O(q^3)\,,\\
h_3(q)&=\tfrac{f(q)-{}^{\sigma}f(q)}{2\sqrt{d}}=O(q^3)\,.
\end{aligned}$$

Case **EEE**. Let be

$$\begin{aligned}
f_1(q)&=q+a_2q^2+O(q^3)\,,\\
f_2(q)&=q+b_2q^2+O(q^3)\,,\\
f_3(q)&=q+c_2q^2+O(q^3)\,.
\end{aligned}$$

Then by (1) we can assume that $a_2\neq b_2$, and thus

$$\begin{aligned}
h_1(q)&=f_1(q)=q+O(q^2)\,,\\
h_2(q)&=\tfrac{f_1(q)-f_2(q)}{a_2-b_2}=q^2+O(q^3)\,,\\
h_3(q)&=f_3(q)-f_1(q)-(c_2-a_2)h_2(q)=O(q^3)\,.
\end{aligned}$$

Let be $n = \text{ord}_q h_3$. For all the cases above, the degree four monomial $h_1^i h_2^j h_3^{4-i-j}$ has order $1+2j+n(4-i-j)$. It is easy to check that for $n \geq 6$ all these orders are different, and hence there is no $F \in \mathbb{Q}[X,Y,Z]$ of degree 4 such that $F(h_1(q), h_2(q), h_3(q)) = 0$.

(ii) Let $S_2(C)$ be choosen as above, then the image $\phi(C)$ of C by the canonical embedding ϕ

$$\phi : q \longmapsto (h_1(q) : h_2(q) : h_3(q))$$

is a smooth plane quartic of the form

$$\sum_{i,j,k \in \mathbb{Z}_{\geq 0}}^{i+j+k=4} a_{ijk} X^i Y^j Z^k = 0\,, \qquad \text{where } a_{ijk} \in \mathbb{Q}.$$

For $X := h_1(q), Y := h_2(q)$ and $Z := h_3(q)$, the degree four monomials $X^i Y^j Z^k$ have q-expansions $X^4 = q^4 + O(q^5)$, $X^3 Y = q^5 + O(q^6)$, and $X^i Y^j Z^k = O(q^6)$ otherwise. By this notation, it follows that $a_{400}, a_{310} = 0$, and therefore $P_\infty := (1:0:0) \in \phi(C)(\mathbb{Q})$. The tangent line l_∞ at P_∞ is the line with equation $Z = 0$. Furthermore for $h_3(q) = q^4 + O(q^5)$, we have $X^2Y^2 = q^6 + O(q^7)$ and $X^i Y^j Z^k = O(q^7)$ for all the degree four monomials different from X^2Y^2, X^4, X^3Y. Hence $a_{220} = 0$. In this case, P_∞ is at least an ordinary flex.

The proof for hyperflex is similar to that just given for previous ones: For $h_3(q) = q^5 + O(q^6)$, the degree four monomials have q-expansions $X^2Y^2 = q^6 + O(q^7)$, $XY^3 = q^7 + O(q^8)$ and $X^i Y^j Z^k = O(q^8)$ for the other degree four monomials different from X^2Y^2, X^4, X^3Y, XY^3. In this case, $a_{220} = a_{130} = 0$, which proves the assertion.

By normalizing the equation of the curve in a appropriate way, it is easy to check that $\psi_F(q) = 1$ since the numerator and denominator of $\psi_F(q)$ have the same order at q and the same first coefficient. □

4.2. **Computational Algorithm.** Proposition 7 and Lemma 3 provide us a theoretical algorithm to recognize if a new modular abelian 3-fold corresponds to a non-hyperelliptic new modular curve of genus 3. The following result give us a computational algorithm:

Proposition 8. *Let A be a modular abelian 3-fold of level N and let $S_2(A) = \langle h_1(q), h_2(q), h_3(q)\rangle_{\mathbb{Q}}$. If there exist a non-singular homogeneous polynomial $F \in \mathbb{Q}[X,Y,Z]$ of degree 4 and a constant $c_F \in \mathbb{Q}^*$ such that*

(i) *$F(h_1(q), h_2(q), h_3(q)) = O(q^{c_N})$, where $c_N = \dfrac{2}{3}[SL_2(\mathbb{Z}) : \Gamma_1(N)]$,*

(ii) *$\Psi_F(q) = c_F + O(q^{c'_N})$, where $c'_N = \dfrac{1}{2}[SL_2(\mathbb{Z}) : \Gamma_1(N)]$,*

then the curve $C : F(X,Y,Z) = 0$ is a non-hyperelliptic modular curve of level N such that $\text{Jac}(C) \overset{\mathbb{Q}}{\sim} A$ and $H^0(A, \Omega^1) = \pi^* H^0(C, \Omega^1)$, where $\pi : X_1(N) \twoheadrightarrow C$ is the corresponding modular parametrization of C.

Remark 3. If A is a quotient of $J_0(N)$ then $\Gamma_1(N)$ could be replaced by $\Gamma_0(N)$ in the formulae of c_N and c'_N. The only case when it could not be replaced is when $A \overset{\mathbb{Q}}{\sim} A_f \times A_g$ and $f \in S_2(\varepsilon, N)$ such that ε is non-trivial. In that case[1], A is a

[1]See [González-Jiménez and González 03, Section 6] for the definition of the congruence subgroup $\Gamma(N,\varepsilon)$ and the corresponding modular curve $X(N,\varepsilon)$.

quotient of $\mathrm{Jac}(X(N,\varepsilon))$ and $\Gamma_1(N)$ could be replaced by $\Gamma(N,\varepsilon)$ in the formulae of c_N and c'_N.

Proof. For a positive integer k, it is well known that if $f \in S_k(N)$ and $f(q) = O(q^c)$ with $c = \frac{k}{12}[SL_2(\mathbb{Z}) : \Gamma_1(N)]$ then $f = 0$. Using condition (i), we apply the above result to the modular form $F(h_1, h_2, h_3) \in S_8(N)$ to prove that $F(h_1(q), h_2(q), h_3(q)) = 0$. Now, using (ii) we are going to prove that $\Psi_F = c_F \in \mathbb{Q}^*$. In this case, we have to be more carefully since the numerator of Ψ_F contains derivative of modular forms which are in general not anymore modular. However, a straightforward computation shows that, if $h_1, h_3 \in S_2(N)$ then $h'_1 h_3 - h_1 h'_3 \in S_6(N)$. Now F_Y is a homogeneous polynomial of degree 3 then $F_Y(h_1, h_2, h_3) \in S_6(N)$. Let $c_F \in \mathbb{Q}^*$ then $G = h'_1 h_3 - h_1 h'_3 - c_F F_Y(h_1, h_2, h_3) \in S_6(N)$. Therefore, $\Psi_F(q) = c_F$ if and only if $G(q) = O(q^{c'_N})$, that is, if and only if (ii) holds. Then the proof of the proposition follows directly from Proposition 7. □

To compute a model of the modular curve C defined over the integers, let $\{h_1, h_2, h_3\}$ be a basis of $S_2(C)$ consisting of cusp forms with integral q-expansions. Let us consider an enumeration

$$\{f_1, \ldots, f_{15}\} = \{h_1^i h_2^j h_3^k \in S_8(N) \,|\, i, j, k \in \mathbb{Z}_{\geq 0},\ i + j + k = 4\}$$

of the set of degree four monomials in h_1, h_2, h_3, and let $f_i(q) = \sum_{j \geq 1} b_{ij} q^j$ be the q-expansions of the f_i. The smooth plane quartic F defining the modular curve C, i.e. with $F(h_1(q), h_2(q), h_3(q)) = 0$, is given by the cusp form $F = \sum_{i=1}^{15} a_i f_i = \sum_{j \geq 1} \left(\sum_{i=1}^{15} b_{ij} a_i\right) q^j \in S_8(N)$, whose defining coefficients a_i are computed by solving the linear equation $B^T \cdot a = 0$, where $a = (a_i)$ is a non-trivial vector with 15 entries, $B = (b_{ij})$ a matrix with 15 rows and at least c_N columns.

In the following example we are going to explain how this method works in practice.

Example 1. Let A be the modular abelian 3-fold A_{243E}. The vector space $S_2(A)$ is generated by the q-integral basis $\{h_1, h_2, h_3\}$, where

$$\begin{aligned} h_1(q) &= q - 3q^5 - 2q^7 - 3q^8 - 2q^{10} + q^{13} + q^{16} - 3q^{17} + 2q^{19} + O(q^{20}), \\ h_2(q) &= q^2 - q^5 - 3q^7 - 4q^8 - 3q^{10} + 4q^{11} + 3q^{13} + 2q^{14} + 3q^{16} + 6q^{19} + O(q^{20}), \\ h_3(q) &= q^4 - 2q^7 - 3q^8 - q^{10} + 3q^{11} + q^{13} + 3q^{14} + 3q^{16} + 3q^{19} + O(q^{20}). \end{aligned}$$

By Proposition 8 we compute the q-expansions of h_1, h_2, h_3 with $c_{203} = 180$ coefficients and then an equation of the modular curve C_{243}^E is

$$C_{243}^E : X^3Z - 3X^2Z^2 - XY^3 + 9XYZ^2 - 6XZ^3 + 2Y^3Z - 9Y^2Z^2 + 9YZ^3 - 2Z^4 = 0.$$

Since $\Psi_F(q) = 1$, the jacobian $\mathrm{Jac}(C_{243}^E)$ is $\mathbb{Q}$-isogenous to A.

4.3. **Automorphisms of new modular curves.** Let C be a new modular curve of level N and genus $g \geq 2$. The diamond operators $\langle d \rangle$ on $X_1(N)$ induce automorphisms of C over $\overline{\mathbb{Q}}$. Let $\mathcal{D}$ be the abelian subgroup of $\mathrm{Aut}_{\overline{\mathbb{Q}}}(C)$ consisting of diamond automorphisms, then $\mathcal{D}$ is $G_{\mathbb{Q}}$-stable. There exists a surjective morphism $X_0(N) \twoheadrightarrow C$ if and only if $\mathcal{D}$ is the trivial group.

Let $\mathcal{D}'$ be a subgroup of $\mathcal{D}$, if the curve $C' = C/\mathcal{D}'$ has genus g', then $g - g'$ is even (see [Baker et al. 05, Lemma 6.17]). In particular, if C is a new modular curve of genus 3 and $\mathcal{D}' \neq \{1\}$, then $g' = 1$. Then if C is non-hyperelliptic, Proposition 6 provides us the following result:

Lemma 4. *Let C be a new modular non-hyperelliptic curve of genus 3. Then $\mathcal{D}$ is either trivial or cyclic of order 2, 3 or 4.*

Corollary 1. *Let C be a new modular non-hyperelliptic genus 3 curve of level N such that $\operatorname{Jac}(C)$ is not a quotient of $J_0(N)$. Then $\operatorname{Jac}(C) \overset{\mathbb{Q}}{\sim} A_f \times A_g$ for some $f, g \in \operatorname{New}_N$ for which the nebentypus of g is trivial and the nebentypus of f is of order 2, 3 or 4.*

5. Non-new non-hyperelliptic modular curves of genus 3

In the case of a non-new modular curve C, we could also use Proposition 7 to compute the equation of C. The only difference is based on the computation of a basis of $S_2(C)$.

Lemma 5. [Baker et al. 05, Lemma 3.5] *Let $\pi : X_1(N) \twoheadrightarrow C$ be a non-constant $\mathbb{Q}$-morphism. The vector space $S_2(C)$ admits a $G_{\mathbb{Q}}$-invariant basis B consisting of cusp forms*

$$h(q) = \sum_{d \mid \frac{N}{M}} c_d f(q^d)$$

for $M|N$, $f \in S_2^{\text{new}}(M)$ and $c_d \in K_f$.

Nevertheless, the above result does not give any bound on the coefficients c_d. The following example shows a non-new non-hyperelliptic modular curve of genus 3.

Example 2. Let A be the non-new non-simple modular abelian 3-fold $A_{178C} \times E_{89A}$. The vector space $S_2(A)$ is generated by $\{f, {}^{\sigma}f, g\}$, where

$$f(q) = q - q^2 + aq^3 + q^4 + (-2a-3)q^5 - aq^6 - 2q^7 - q^8 + O(q^9) \in \operatorname{New}_{178},$$
$$g(q) = q - q^2 - q^3 - q^4 - q^5 + q^6 - 4q^7 + 3q^8 - 2q^9 + q^{10} + O(q^{11}) \in \operatorname{New}_{89},$$

and $K_f = \mathbb{Q}(a)$, $a^2 + 2a - 1 = 0$, $\operatorname{Aut}(K_f) = \{\operatorname{id}, \sigma\}$ and $K_g = \mathbb{Q}$. Let $\{f_1, f_2\}$ be the $\mathbb{Q}$-basis of $S_2(A_{178C})$ with the following q-expansion:

$$f_1(q) = q - q^2 + q^4 - 3q^5 - 2q^7 - q^8 - 2q^9 + 3q^{10} - 2q^{13} + 2q^{14} + O(q^{15}),$$
$$f_2(q) = q^3 - 2q^5 - q^6 - 2q^9 + 2q^{10} + 2q^{11} + q^{12} + q^{15} + 2q^{17} + 2q^{18} + O(q^{19}).$$

There exists a smooth plane quartic $F(X, Y, Z) = 0$ such that

$$F(f_1(q), f_2(q), g(q) + 2g(q^2)) = 0,$$

namely the curve C_{178C}^{89A} with equation

$$F(X,Y,Z) = X^4 - 8X^3Y + 38X^2Y^2 - 2X^2Z^2 - 24XY^3 - 8XYZ^2 - 7Y^4 + 6Y^2Z^2 + Z^4.$$

Then C_{178C}^{89A} is a non-hyperelliptic and non-new modular curve of genus 3. With $f_3(q) = g(q) + 2g(q^2)$, the use of the normalized basis

$$h_1 = f_2 + f_3, \quad h_2 = \frac{1}{2}(f_2 + f_3 - f_1), \quad h_3 = f_2$$

gives an equation for C_{178C}^{89A} with smaller integer coefficients

$$\begin{aligned} G(X,Y,Z) \;=\; & X^3Z - X^2Y^2 - 3X^2YZ - 4X^2Z^2 + 2XY^3 + 5XY^2Z \\ & + 11XYZ^2 + 3XZ^3 - Y^4 - 4Y^3Z - 9Y^2Z^2 - 4YZ^3 \end{aligned}$$

with $G(h_1(q), h_2(q), h_3(q)) = 0$ and $\psi_G(q) = 1$.

The following couple of examples show (plane) non-hyperelliptic curves of genus 3 with equations $C_d : F_d(X, Y, Z) = 0$ of degree $d \geq 5$.

Example 3. Let A be the modular abelian 3-fold A_{178D}. With respect to the integral basis $\{f_1, f_2, f_3\}$ of $S_2(A)$

$$\begin{aligned} f_1(q) &= q + q^2 + q^4 + q^8 + 3q^9 + 2q^{11} - 6q^{15} + q^{16} - 2q^{17} + 3q^{18} - 4q^{19} + O(q^{20}) \,, \\ f_2(q) &= q^3 - q^5 + q^6 - q^9 - q^{10} + q^{12} - 2q^{13} + q^{15} + q^{17} - q^{18} + q^{19} + O(q^{20}) \,, \\ f_3(q) &= q^7 - 2q^9 - q^{13} + q^{14} + 2q^{15} + 2q^{17} - 2q^{18} + O(q^{20}) \,, \end{aligned}$$

we compute the equation of a genus 3 non-hyperelliptic curve $C \,:\, F_7(X, Y, Z) = 0$, where

$$\begin{aligned} F_7(X,Y,Z) = {} & X^5Z^2 - 3X^4YZ^2 + 8X^4Z^3 - 2X^3Y^3Z + 7X^3Y^2Z^2 - 23X^3YZ^3 \\ & +26X^3Z^4 - 3X^2Y^3Z^2 + 18X^2Y^2Z^3 - 53X^2YZ^4 + 42X^2Z^5 + XY^6 \\ & -3XY^5Z - Y^7 + XY^4Z^2 + 14XY^3Z^3 - 10XY^2Z^4 - 36XYZ^5 + 32XZ^6 \\ & +4Y^6Z + 10Y^5Z^2 - 66Y^4Z^3 + 124Y^3Z^4 - 100Y^2Z^5 + 20YZ^6 + 8Z^7 \,, \end{aligned}$$

for which $F_7(f_1(q), f_2(q), f_3(q)) = 0$. The canonical embedding of C is computed using Magma, and $C' = \phi(C)$ has a model with the equation

$$\begin{aligned} C' \;:\; & X^4 - 4X^3Y + 6X^3Z + 2X^2Y^2 - 5X^2Z^2 + 4XY^3 - 30XY^2Z \\ & +64XYZ^2 - 42XZ^3 - 3Y^4 + 8Y^3Z + 9Y^2Z^2 - 40YZ^3 + 28Z^4 = 0 \,. \end{aligned}$$

We have been able to check $\mathrm{Jac}(C') \overset{\mathbb{F}_p}{\sim} \mathrm{Jac}(C^{89A}_{178C})$ from Example 2 for $p \nmid 178$ such that $p < 500$. Furthermore by computing the absolute Dixmier-Ohno invariants [Dixmier 87], we have $C' \overset{\overline{\mathbb{Q}}}{\simeq} C^{89A}_{178C}$.

Starting with a $\mathbb{Q}$-simple abelian variety $A = A_f$, the jacobian $\mathrm{Jac}(C')$ of the above curve C' and A_f are in general not $\overline{\mathbb{Q}}$-isogenous. However, if $\mathrm{Jac}(C')$ is $\mathbb{Q}$-simple and new of level N, then there exists a newform $g \in \mathrm{New}_N$ with $\mathrm{Jac}(C') \overset{\mathbb{Q}}{\sim} A_g$. If we also have $g = f \otimes \chi$ where χ is a Dirichlet-character then $A_f \overset{K}{\sim} \mathrm{Jac}(C')$, where $K = \overline{\mathbb{Q}}^{\ker \chi}$ ([Shimura 73]).

Example 4. Let A be the modular abelian 3-fold A_{243F}. Then there exists an integral basis $\{f_1, f_2, f_3\}$ of $S_2(A)$ satisfying $F_6(f_1(q), f_2(q), f_3(q)) = 0$, where

$$\begin{aligned} F_6(X,Y,Z) = {} & X^5Z - 7X^4Z^2 - X^3Y^3 - 9X^3YZ^2 + X^3Z^3 + 6X^2Y^3Z + 9X^2Y^2Z^2 \\ & +27X^2YZ^3 + 19X^2Z^4 - 3XY^3Z^2 + 18XY^2Z^3 + 27XYZ^4 + 2XZ^5 \\ & +8Y^3Z^3 + 9Y^2Z^4 - 9YZ^5 - 8Z^6 \,, \end{aligned}$$

defines a non-hyperelliptic plane curve $C \,:\, F_6(X, Y, Z) = 0$ of genus 3 and degree $d = 6$. The canonical embedding of C is a smooth plane quartic C' given by

$$\begin{aligned} C' \;:\; & X^3Y - 2X^3Z - 12X^2Y^2 + 9X^2YZ - 24X^2Z^2 + 48XY^3 + 24XY^2Z \\ & -57XYZ^2 + 66XZ^3 - 64Y^4 + 104Y^3Z - 36Y^2Z^2 - 65YZ^3 + 88Z^4 = 0 \,. \end{aligned}$$

We have $f_{243E} = f_{243F} \otimes \chi$ where χ is the non-trivial Dirichlet-character of level 3, then $A_{243E} \overset{\mathbb{Q}(\sqrt{-3})}{\sim} A_{243F}$. In fact, the quartic C' is modular and its jacobian $\mathrm{Jac}(C')$ is conjecturaly $\mathbb{Q}$-isogenous to $A_{243E} \overset{\mathbb{Q}}{\sim} \mathrm{Jac}(C^E_{243})$ (we have checked $\mathrm{Jac}(C') \overset{\mathbb{F}_p}{\sim} A_{243E}$ for primes p with $3 < p < 500$).

The next example shows that condition (ii) in Proposition 8 is necessary.

Example 5. Let $A_1 = A_{120A_{\{0,0,0,2\}}} \times E_{120B}$ where $H^0(A_{120A_{\{0,0,0,2\}}}, \Omega^1) = \langle f_1(q), f_2(q)\rangle\, dq/q$ and $H^0(E_{120B}, \Omega^1) = \langle f_3(q)\rangle\, dq/q$ are some integral basis of the associated space of holomorphic differentials.

For the curve $C_1 : F(X, Y, Z) = 0$ given by

$$F(X, Y, Z) = 3\,X^4 - 10\,X^2\,Y^2 - 10\,X^2\,Z^2 + 7\,Y^4 + 8\,Y^3\,Z + 2\,Y^2\,Z^2 - 8\,Y\,Z^3 + 7\,Z^4$$

we have $F(f_1(q), f_2(q), f_3(q)) = 0$ but $\psi_F(q) \notin \mathbb{C}$.

Let $A_2 = A_{240B_{\{0,0,0,2\}}} \times E_{240A}$ where $H^0(A_{240B_{\{0,0,0,2\}}}, \Omega^1) = \langle h_1(q), h_2(q)\rangle\, dq/q$ and $H^0(E_{240A}, \Omega^1) = \langle h_3(q)\rangle\, dq/q$ are some integral basis of the associated space of holomorphic differentials.

For the curve $C_2 : G(X, Y, Z) = 0$ given by

$$G(X, Y, Z) = 3\,X^4 - 10\,X^2\,Y^2 - 10\,X^2\,Z^2 + 7\,Y^4 - 8\,Y^3\,Z + 2\,Y^2\,Z^2 + 8\,Y\,Z^3 + 7\,Z^4$$

we have $G(h_1(q), h_2(q), h_3(q)) = 0$ but $\psi_G(q) \notin \mathbb{C}$.

Furthermore, $F(X, -Y, Z) = G(X, Y, Z)$, and hence $Jac(C_1) \overset{\mathbb{Q}}{\simeq} Jac(C_2)$.

However A_1 is not $\mathbb{Q}$-isogenous to A_2. In fact, if we denote by χ_1 (resp. χ_2) the Dirichlet character attached to the quadratic field $\mathbb{Q}(\sqrt{-5})$ (resp. $\mathbb{Q}(\sqrt{-1})$), the following equalities hold

$$f_{120A_{\{0,0,0,2\}}} = f_{240B_{\{0,0,0,2\}}} \otimes \chi_1 \quad \text{and} \quad f_{120B} = f_{240A} \otimes \chi_2.$$

Therefore

$$A_{120A_{\{0,0,0,2\}}} \overset{\mathbb{Q}(\sqrt{-5})}{\sim} A_{240B_{\{0,0,0,2\}}} \quad \text{and} \quad E_{120B} \overset{\mathbb{Q}(\sqrt{-1})}{\sim} E_{240A}.$$

Furthermore, $Jac(C_i)$ is not $\overline{\mathbb{Q}}$-isogenous to the modular 3-fold A_i for $i = 1, 2$. Nevertheless, $C_1 \overset{\mathbb{Q}}{\simeq} C_2$ must be modular for some level M dividing 120. Indeed, C_1 is $\mathbb{Q}$-isomophic to a non-new modular curve C of level 30 with $Jac(C) \overset{\mathbb{Q}}{\sim} A_{30A_{\{0,2\}}} \times E_{15A}$. More precisely, using an integral basis of $H^0(A_{30A_{\{0,2\}}}, \Omega^1) = \langle g_1(q), g_2(q)\rangle\, dq/q$ and $H^0(E_{15A}, \Omega^1) = \langle f(q)\rangle\, dq/q$ and applying the canonical map to the basis of regular differentials $H^0(A_{30A_{\{0,2\}}} \times E_{15A}, \Omega^1) = \langle g_1(q), g_2(q), f(q) + 2f(q^2)\rangle\, dq/q$ we obtain

$$H(g_1(q), g_2(q), f(q) + 2f(q^2)) = 0$$

and $\Psi_H(q) \in \mathbb{Q}^*$ where the curve $C : H(X, Y, Z) = 0$ given by

$$H(X, Y, Z) = 7X^4 + 8X^3Y + 2X^2Y^2 - 10X^2Z^2 - 8XY^3 + 7Y^4 - 10Y^2Z^2 + 3Z^4$$

is a smooth plane quartic with $F(Z, Y, -X) = H(X, Y, Z)$.

Conclusion

In this work we present a method to compute equations of modular non-hyperelliptic curves of genus 3. It should be pointed out that the problem of computing the complete list of new modular non-hyperelliptic (or hyperelliptic) curves of genus 3 is still an open question.

We also stress the following open question:

For every fixed genus $g \geq 2$, is the number of modular curves (without the restriction new) of genus g infinite?

The authors of [Baker et al. 05] conjectured that the answer of the above question is *yes.*

As first step to answer the above question, we explained by means of some specific example the difference that may appear between new and non-new modular curves of genus 3.

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Appendix

Performing the computations of all the non-hyperelliptic new modular curves of genus 3 would be extremely time-consuming. Therefore we have conducted a search of all non-hyperelliptic new modular curves of genus 3 and some fixed level. For the case **A** until level $N \leq 10000$, for the case **AE** until level $N \leq 4000$ and for the case **EEE** until level $N \leq 130000$ (the highest level in Cremona's tables [Cremona 06]). For this aim, we have implemented the method developed at section 4 in Magma [Bosma et al. 97] using W. A. Stein's Modular packages. Then we have obtained a total of 44 such curves that appear at Table 1.

Table 1 has two columns. The first one shows the label of the non-hyperelliptic new modular curve of genus 3 and the second column shows the correponding smooth plane quartic model over $\mathbb{Z}$. We have used the labelling of modular abelian varieties and Dirichlet characters as it was introduced at [Baker et al. 05]. The notation of the curves is as follows:

Notation. Let C be a new modular non-hyperelliptic genus 3 curve of level N. In the case that $\mathrm{Jac}(C)$ is a factor of $J_0(N)$ we will add N as a subscript and the corresponding letters of the labels corresponding to the $\mathbb{Q}$-factors of $\mathrm{Jac}(C)$ as superscripts to C. Otherwise, $\mathrm{Jac}(C) \overset{\mathbb{Q}}{\sim} A_f \times E$, where $f \in S_2(N, \varepsilon)$ such that ε is not trivial and E is an elliptic curve over $\mathbb{Q}$. Then we will denote this modular curve by $C_{N\,X_{A_\varepsilon}}^{X_E}$ where X_{A_ε} and X_E are the corresponding labels of A_f and E respectively.

Finally, we have added the superscript ♦ (resp. ★) on the left of the labelling of the curve if P_∞ is an ordinary flex (resp. hyperflex).

Note that we have not attempted to reduce the coefficients appearing in the computed models, however the models obtained have already very small coefficients: The worst-case is the modular curve $C_{65}^{A,B}$, which has largest coefficient 98.

Contrary to what is observed in [Baker et al. 05] for hyperelliptic modular curves of genus 3, there exists a non-hyperelliptic new modular curve of genus 3 and level N for which N has more than two different odd prime divisors, namely the modular curve C_{855}^{H}. In fact, this is the only curve for which three different primes appear in the factorization of the level of modularity. In the rest of the cases, there are just one or two primes.

As a final remark, note that the curve $C_{49,A_{\{14\}}}^{A}$ has the same equation that the Klein quartic which is $\mathbb{Q}$-isomorphic to the classical modular curve $X(7)$.

Table 1: New modular non-hyperelliptic curves of genus 3

C	:	$F(x,y,z) = 0$
$C_{20,A_{\{1,1\}}}^{A}$	:	$x^3z - x^2y^2 - 3x^2z^2 + xy^3 + 4xz^3 - 2z^4 = 0$
$C_{24,A_{\{0,1,0\}}}^{A}$	:	$x^3z - x^2y^2 - x^2z^2 + xy^3 - xy^2z - 3xyz^2 + y^3z + 2y^2z^2 + yz^3 = 0$
${}^{\blacklozenge}C_{24,A_{\{1,1,1\}}}^{A}$	:	$x^3z - 2x^2yz - x^2z^2 - xy^3 + 2xy^2z + 6xyz^2 + 2y^3z - 2y^2z^2 - 4yz^3 = 0$
${}^{\blacklozenge}C_{36,A_{\{1,3\}}}^{A}$	:	$x^3z - 3x^2z^2 - xy^3 + 4xz^3 + 2y^3z - 2z^4 = 0$
${}^{\bigstar}C_{39,A_{\{0,6\}}}^{A}$	:	$x^3z - 2x^2z^2 + 4xy^2z - 7xyz^2 - 6xz^3 - y^4 + 5y^3z + 2y^2z^2 - 6yz^3 - 3z^4 = 0$
${}^{\blacklozenge}C_{39,A_{\{0,4\}}}^{A}$	:	$x^3z - 2x^2yz - xy^3 - 2xy^2z + 2xyz^2 + yz^3 = 0$
$C_{43}^{A,B}$	:	$2x^3z - 2x^2y^2 - 6x^2z^2 + xy^3 + 9xy^2z - 5xyz^2 + 11xz^3 - 9y^4 + 12y^3z - 22y^2z^2 + 12yz^3 - 9z^4 = 0$
${}^{\blacklozenge}C_{45,A_{\{2,0\}}}^{A}$	:	$x^3z + 2x^2yz - xy^3 + 2xy^2z - 2xyz^2 + yz^3 = 0$
${}^{\blacklozenge}C_{49,A_{\{14\}}}^{A}$	:	$x^3z - xy^3 + yz^3 = 0$
${}^{\blacklozenge}C_{56,A_{\{0,1,0\}}}^{A}$	:	$x^3z + 2x^2yz - x^2z^2 - xy^3 - 2xy^2z - 6xyz^2 + 2y^3z + 2y^2z^2 + 4yz^3 = 0$
$C_{57}^{A,B,C}$	:	$2x^3z - 2x^2y^2 + 5x^2z^2 - 16xy^2z - 8xyz^2 + 2xz^3 + 3y^4 + 8y^3z - 6y^2z^2 - 4yz^3 = 0$
$C_{65}^{A,B}$	:	$2x^3z - 2x^2y^2 - 7x^2z^2 - 2xy^3 - 4xy^2z + 26xyz^2 + 30xz^3 - 3y^4 - 26y^3z - 81y^2z^2 - 98yz^3 - 40z^4 = 0$
$C_{65}^{A,C}$	:	$6x^3z - 6x^2y^2 - 8x^2z^2 - 3xy^3 + 25xy^2z - 13xyz^2 + 25xz^3 - 11y^4 + 19y^3z - 33y^2z^2 + 13yz^3 - 14z^4 = 0$
$C_{82}^{A,B}$	:	$x^3z - x^2y^2 - 2x^2z^2 + 4xy^2z + 3xyz^2 + y^3z - 2yz^3 = 0$

C	:	$F(x,y,z) \ = \ 0$
$C_{91}^{A,C}$	:	$x^3z-x^2y^2-x^2z^2+xy^3-xy^2z+3xyz^2-xz^3-2y^4+4y^3z-6y^2z^2+4yz^3-z^4=0$
C_{97}^{A}	:	$x^3z-x^2y^2-5x^2z^2+xy^3+xy^2z+3xyz^2+6xz^3-3y^2z^2-yz^3-2z^4=0$
$\bigstar C_{99}^{A,C,D}$	:	$x^3z-3x^2z^2+3xy^2z-3xyz^2+9xz^3-y^4-6y^2z^2+yz^3-8z^4=0$
$\blacklozenge C_{109}^{B}$	:	$x^3z-2x^2yz-x^2z^2-xy^3+6xy^2z-6xyz^2+3xz^3+y^4-6y^3z+10y^2z^2-5yz^3=0$
C_{113}^{C}	:	$x^3z-x^2y^2-4x^2z^2+xy^3+2xy^2z+6xz^3-y^3z-3y^2z^2+yz^3-3z^4=0$
$C_{118}^{A,C,D}$	:	$x^3z-x^2y^2-x^2z^2+2xy^2z+xyz^2+xz^3+y^3z+y^2z^2+yz^3+z^4=0$
$C_{123}^{A,C}$	:	$x^3z-x^2y^2+x^2z^2-xy^3-2xy^2z+xz^3-y^4-y^3z-y^2z^2=0$
C_{127}^{A}	:	$x^3z-x^2y^2-3x^2z^2+xy^3-xyz^2+4xz^3+2y^3z-3y^2z^2+3yz^3-2z^4=0$
C_{139}^{B}	:	$x^3z-x^2y^2-2x^2z^2+xy^3-2xy^2z+2xyz^2+xz^3+y^4-2y^3z+4y^2z^2-3yz^3=0$
$C_{141}^{A,C,D}$	:	$x^3z-x^2y^2+x^2z^2-xy^3+xy^2z+xz^3-y^4-y^3z-y^2z^2=0$
C_{149}^{A}	:	$x^3z-x^2y^2-3x^2z^2+xy^3+3xy^2z-2xyz^2+2xz^3-y^4-y^2z^2+yz^3=0$
$\blacklozenge C_{151}^{A}$	:	$x^3z-2x^2yz-2x^2z^2-xy^3+2xy^2z+4xyz^2+xz^3+y^2z^2-3yz^3-2z^4=0$
C_{169}^{B}	:	$x^3z-x^2y^2-3x^2z^2+xy^3+2xyz^2+xz^3+y^2z^2-3yz^3+z^4=0$
$\blacklozenge C_{179}^{B}$	:	$x^3z-2x^2yz-2x^2z^2-xy^3+2xy^2z+xyz^2+2xz^3+y^2z^2-yz^3-z^4=0$
C_{187}^{E}	:	$x^3z-x^2y^2-x^2z^2+xy^3-xy^2z-xyz^2+2xz^3+y^3z-y^2z^2+3yz^3=0$
C_{203}^{F}	:	$x^3z-x^2y^2-3x^2z^2+xy^3+3xy^2z-4xyz^2+4xz^3-y^4+3y^3z-6y^2z^2+3yz^3-2z^4=0$
C_{217}^{A}	:	$3x^3z-3x^2y^2-11x^2z^2-3xy^3+13xy^2z-2xyz^2+11xz^3-2y^4-y^3z-4y^2z^2+yz^3-2z^4=0$
C_{239}^{A}	:	$x^3z-x^2y^2-x^2z^2+xy^3-xy^2z+xz^3+y^4-y^3z+yz^3-z^4=0$
$\blacklozenge C_{243}^{E}$	:	$x^3z-3x^2z^2-xy^3+9xyz^2-6xz^3+2y^3z-9y^2z^2+9yz^3-2z^4=0$
$\blacklozenge C_{243}^{A,D}$	:	$x^3z-xy^3+6xz^3-4y^3z+7z^4=0$
C_{295}^{A}	:	$x^3z-x^2y^2-x^2z^2+xy^3-xy^2z+2xyz^2-xz^3-y^3z+3y^2z^2-yz^3=0$
C_{329}^{C}	:	$x^3z-x^2y^2+xy^3+xyz^2+xz^3-y^3z+2yz^3+z^4=0$
$\blacklozenge C_{369}^{D}$	:	$x^3z-2x^2z^2-xy^3+6xyz^2-6xz^3-3y^2z^2+6yz^3-z^4=0$
$\blacklozenge C_{459}^{B,I}$	:	$x^3z-x^2z^2-xy^3+5xyz^2-xz^3+y^4+2y^3z-y^2z^2-2yz^3=0$
C_{475}^{E}	:	$x^3z-x^2y^2-5x^2z^2-xy^3+xy^2z+17xyz^2+14xz^3-2y^4-14y^3z-35y^2z^2-35yz^3-12z^4=0$
$\blacklozenge C_{855}^{H}$	:	$x^3z-x^2z^2-xy^3+3xyz^2-3xz^3+2y^3z-3y^2z^2+3yz^3=0$
C_{1175}^{D}	:	$x^3z-x^2y^2+x^2z^2+xy^3-2xy^2z+2xyz^2-xz^3+y^4-2y^3z+yz^3=0$
$\blacklozenge C_{1215}^{P}$	:	$x^3z-xy^3+3xyz^2+5xz^3-6y^2z^2-3yz^3+z^4=0$
$\blacklozenge C_{1215}^{A,K}$	:	$x^3z-xy^3+3xyz^2+5xz^3+3y^2z^2+6yz^3-8z^4=0$
$\bigstar C_{1539}^{C,D,E}$	:	$x^3z-3x^2z^2+3xy^2z-3xyz^2+3xz^3-y^4-2y^2z^2+yz^3+2z^4=0$

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