New Sequence Families with Zero or Low Correlation Zone via Interleaving Techniques

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Abstract

Sequence families with zero or low correlation zone can be used in the quasi-synchronous code-division multiple-access (QS-CDMA) communication systems. Interleaving techniques are very useful for sequence design. In this paper, we present a general construction of sequence families with zero or low correlation zone using interleaving techniques and complex Hadamard matrices. The component sequences are perfect or ideal two-level. In two cases, we construct the shift sequences: 1) $P|L$; 2) P is even, and $L \equiv P/2 \pmod{P}$. The conditions are derived under which the new construction is optimal. Some examples are also given to specify the new construction.

Index Terms. Quasi-synchronous code-division multiple access (QS-CDMA), low correlation zone (LCZ) sequence, zero correlation zone (LCZ) sequence, interleaving technique.

1 Introduction

In 1992, Gaudenzi, Elia, and Viola proposed the quasi-synchronous code-division multiple-access (QS-CDMA) communication systems [1]. Unlike the conventional code-division multiple-access (CDMA) systems [2], a time delay between the signals of different users within a few chips is allowed in QS-CDMA systems. There have been some interesting developments involving QS-CDMA communication systems recently. Sequences with low correlation property or zero correlation property around the origin can be used in such systems for reducing multiple-access interference [8]. Such sequence set is called low correlation zone (LCZ) or zero correlation zone (ZCZ) sequence. Because LCZ or ZCZ sequences have smaller correlation magnitude within the zone, they show better performance than other well known sequence families with optimal correlation magnitude [8].

Long, Zhang, and Hu constructed a binary LCZ sequence set using GMW sequences [8]. Following the idea of Long *et al*, Tang and Fan proposed p-ary LCZ sequences [9]. Torii, Nakamura, and Suehiro proposed two methods for constructing ZCZ sequences based on perfect sequences and unitary matrices [11]. For quaternary case, Kim, Jang, No, and Chung proposed some optimal quaternary LCZ sequence sets [6]. Later, they designed some optimal or almost optimal LCZ sequences in both binary and nonbinary cases [7]. Jang, No, Chung, and Tang designed some optimal p-ary LCZ sequences [5]. In 2007, Gong, Golomb, and Song described a nice general approach to the construction of LCZ sequences using sequences with subfield decompositions [4]. Some previous known LCZ sequences can be obtained easily from this general construction. Based on the interleaving technique which can be realized as $M \times 2$ arrays, Zhou, Tang, and Gong designed a class of ZCZ or LCZ sequences recently [12]. These sequences are optimal or almost optimal.

In this paper, we present a general construction of LCZ or ZCZ sequences based on interleaving techniques. In many cases, the new sequence sets are optimal with respect to the theoretical bound of Tang, Fan, and Matsufuji which is from Welch bound [10]. The construction in [12] is just a special case of this new construction.

This paper is organized as follows. In Section 2, we give some notation and background which will be needed. In Section 3, we present the new construction. We construct the shift sequences and derive the conditions under which they are optimal in Sections 4 and 5. Finally, Section 6 concludes this paper.

2 Preliminaries

Let $\mathbf{s} = \{s_i\}$ and $\mathbf{t} = \{t_i\}$ be two complex-valued sequences with period N. Then the crosscorrelation $C_{\mathbf{s},\mathbf{t}}(\tau)$ between $\mathbf{s} = \{s_i\}$ and $\mathbf{t} = \{t_i\}$ at shift τ is defined by

$$C_{\mathbf{s},\mathbf{t}}(\tau) = \sum_{i=0}^{N-1} s_i t_{i+\tau}^*, \ 0 \leq \tau < N,$$

where $t_{i+\tau}^*$ is the complex conjugate of $t_{i+\tau}$. If $\mathbf{s} = \mathbf{t}$, then $C_{\mathbf{s},\mathbf{t}}(\tau)$ is called the autocorrelation of $\mathbf{s}$. In this case, we denote it by $C_{\mathbf{s}}(\tau)$ for simplicity.

Definition 1 *A sequence* $\mathbf{s} = \{s_i\}$ *with period* N *is called a perfect sequence if* $C_{\mathbf{s}}(\tau) = 0$ *for any* $0 < \tau < N$.

Definition 2 *A sequence* $\mathbf{s} = \{s_i\}$ *with period* N *is called an ideal two-level sequence if* $C_{\mathbf{s}}(\tau) = -1$ *for any* $0 < \tau < N$.

Definition 3 *Let* $\mathbf{s}_i = \{s_{i,0}, s_{i,1}, ..., s_{i,N-1}\}, 0 \le i < M$, *be* M *shift-distinct complex-valued sequences with period* N. *The set* $\mathcal{S}$ *is called an LCZ sequence set with parameters* (N, M, L, δ) *if*

$$|C_{\mathbf{s}_i,\mathbf{s}_j}(\tau)| \le \delta \text{ for any } |\tau| < L, 0 \le i \ne j < M,$$

and

$$|C_{\mathbf{s}_i}(\tau)| \le \delta \text{ for any } 0 < |\tau| < L, 0 \le i < M.$$

If $\delta = 0$, *then* $\mathcal{S}$ *is called a ZCZ sequence set with parameters* (N, M, L).

In 1995, Gong proposed the concept of interleaved sequences [3]. From then on, the interleaving technique is an important tool for sequence design.

Given a sequence $\mathbf{s} = \{s_0, s_1, ..., s_{N-1}\}$ with period N, a sequence $\boldsymbol{\omega} = \{\omega_0, \omega_1, ..., \omega_{P-1}\}$ of length P with $|\omega_i| = 1, 0 \le i < P$, and a shift sequence $\mathbf{e} = \{e_0, e_1, ..., e_{P-1}\}$, where $0 \le e_i < N, 0 \le i < P$, we can construct a sequence $\mathbf{a} = \{a_0, a_1, ..., a_{NP-1}\}$ with period NP using the interleaving technique. Namely,

$$a_{iP} = \omega_0 s_{i+e_0},\ a_{iP+1} = \omega_1 s_{i+e_1}, ...,\ a_{iP+P-1} = \omega_{P-1} s_{i+e_{P-1}}$$

for $i = 0, 1, ..., N-1$. The new sequence $\mathbf{a}$ can also be written in the following matrix form:

$$\mathbf{a} = \begin{bmatrix} \omega_0 s_{e_0} & \omega_1 s_{e_1} & \cdots & \omega_{P-1} s_{e_{P-1}} \\ \omega_0 s_{1+e_0} & \omega_1 s_{1+e_1} & \cdots & \omega_{P-1} s_{1+e_{P-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0 s_{N-1+e_0} & \omega_1 s_{N-1+e_1} & \cdots & \omega_{P-1} s_{N-1+e_{P-1}} \end{bmatrix}.$$

We denote $\mathbf{a}$ as $\mathbf{a}(\mathbf{s}, \boldsymbol{\omega}, \mathbf{e})$, and it is referred to as an interleaved sequence associated with $(\mathbf{s}, \boldsymbol{\omega}, \mathbf{e})$.

Let L be the left cyclic shift operator. Then for any $\tau = P\tau_1$, we have

$$L^\tau(\mathbf{a}) = \begin{bmatrix} \omega_0 s_{\tau_1+e_0} & \omega_1 s_{\tau_1+e_1} & \cdots & \omega_{P-1} s_{\tau_1+e_{P-1}} \\ \omega_0 s_{\tau_1+1+e_0} & \omega_1 s_{\tau_1+1+e_1} & \cdots & \omega_{P-1} s_{\tau_1+1+e_{P-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_0 s_{\tau_1+N-1+e_0} & \omega_1 s_{\tau_1+N-1+e_1} & \cdots & \omega_{P-1} s_{\tau_1+N-1+e_{P-1}} \end{bmatrix},$$

and for any $\tau = P\tau_1 + \tau_0$ with $0 < \tau_0 < P$, we have

$$L^\tau(\mathbf{a}) = \begin{bmatrix} \omega_{\tau_0} s_{\tau_1+e_{\tau_0}} & \cdots & \omega_{\tau_{P-1}} s_{\tau_1+e_{\tau_{P-1}}} & \omega_0 s_{\tau_1+1+e_0} & \cdots & \omega_{\tau_0-1} s_{\tau_1+1+e_{\tau_0-1}} \\ \omega_{\tau_0} s_{\tau_1+1+e_{\tau_0}} & \cdots & \omega_{\tau_{P-1}} s_{\tau_1+1+e_{\tau_{P-1}}} & \omega_0 s_{\tau_1+2+e_0} & \cdots & \omega_{\tau_0-1} s_{\tau_1+2+e_{\tau_0-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ \omega_{\tau_0} s_{\tau_1+N-1+e_{\tau_0}} & \cdots & \omega_{\tau_{P-1}} s_{\tau_1+N-1+e_{\tau_{P-1}}} & \omega_0 s_{\tau_1+e_0} & \cdots & \omega_{\tau_0-1} s_{\tau_1+e_{\tau_0-1}} \end{bmatrix}.$$

Let $\mathbf{b} = \mathbf{b}(\mathbf{s}, \boldsymbol{\omega}, \mathbf{f})$ be another interleaved sequence associated with $(\mathbf{s}, \boldsymbol{\omega}, \mathbf{f})$, where $\mathbf{f} = \{f_0, f_1, ..., f_{P-1}\}$ with $0 \leq f_i < N, 0 \leq i < P$. Then the crosscorrelation $C_{\mathbf{a},\mathbf{b}}(\tau)$ between $\mathbf{a}$ and $\mathbf{b}$ at shift τ can be written as

$$C_{\mathbf{a},\mathbf{b}}(\tau) = \begin{cases} \sum_{i=0}^{P-1} C_{\mathbf{s}}(\tau_1 + f_i - e_i), & \text{if } \tau = P\tau_1; \\ \sum_{i=0}^{P-1-\tau_0} \omega_i \omega_{i+\tau_0}^* C_{\mathbf{s}}(\tau_1 + f_{i+\tau_0} - e_i) + \sum_{i=P-\tau_0}^{P-1} \omega_i \omega_{i+\tau_0}^* C_{\mathbf{s}}(\tau_1 + 1 + f_{i+\tau_0} - e_i), & \\ \qquad \text{if } \tau = P\tau_1 + \tau_0 \text{ with } 0 < \tau_0 < P, & \end{cases} \tag{1}$$

where the index is reduced by modulo P. For a set or vector $\mathcal{V}$ whose elements are taken from $\mathbb{Z}_N = \{0, 1, ..., N-1\}$, we denote the minimal element in $\mathcal{V}$ by $\min(\mathcal{V})$.

Let

$$\mathcal{A} = \{e_0 - f_0, e_1 - f_1, ..., e_{P-1} - f_{P-1}\},$$

$$\mathcal{B} = \{e_0 - f_{\tau_0}, e_1 - f_{\tau_0+1}, ..., e_{P-1-\tau_0} - f_{P-1}\},$$

and

$$\mathcal{C} = \{e_{P-\tau_0} - f_0 - 1, e_{P-\tau_0+1} - f_1 - 1, ..., e_{P-1} - f_{\tau_0-1} - 1\},$$

where the elements of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are in $\mathbb{Z}_N$. Suppose that the nontrivial autocorrelation of $\mathbf{s}$ is upper bounded by δ, i.e., $|C_{\mathbf{s}}(\tau)| \leq \delta$ for any $0 < \tau < N$. By (1), if $\tau_1 < \min(\mathcal{A} \cup \mathcal{B} \cup \mathcal{C})$, then $|C_{\mathbf{a},\mathbf{b}}(\tau)| \leq P\delta$. In particular, if $\delta = 0$, then $|C_{\mathbf{a},\mathbf{b}}(\tau)| = 0$.

From (1), if

$$\omega_0 \omega_{\tau_0}^* = \omega_1 \omega_{\tau_0+1}^* = ... = \omega_{P-1} \omega_{\tau_0-1}^* \tag{2}$$

holds for any $0 < \tau_0 < P$, then $\mathbf{a}$ and $\mathbf{b}$ are shift equivalent if and only if

$$e_0 - f_0 = e_1 - f_1 = ... = e_{P-1} - f_{P-1}$$

or

$$e_0 - f_{\tau_0} = ... = e_{P-1-\tau_0} - f_{P-1} = e_{P-\tau_0} - f_0 - 1 = ... = e_{P-1} - f_{\tau_0-1} - 1.$$

Otherwise, $\mathbf{a}$ and $\mathbf{b}$ are shift equivalent if and only if

$$e_0 - f_0 = e_1 - f_1 = ... = e_{P-1} - f_{P-1}.$$

Let $\mathbf{c} = \mathbf{c}(\mathbf{s}, \boldsymbol{\rho}, \mathbf{g})$ be another sequence of period NP associated with $\boldsymbol{\rho} = (\rho_0, \rho_1, ..., \rho_{P-1})$ with $|\rho_i| = 1, 0 \leq i < P$, and the shift sequence $\mathbf{g} = \{g_0, g_1, ..., g_{P-1}\}$ with $0 \leq g_i < N, 0 \leq i < P$. Namely,

$$c_{iP} = \rho_0 s_{i+e_0},\ c_{iP+1} = \rho_1 s_{i+e_1}, ...,\ c_{iP+P-1} = \rho_{P-1} s_{i+e_{P-1}}$$

for $i = 0, 1, ..., N-1$. Then the crosscorrelation $C_{\mathbf{a},\mathbf{c}}(\tau)$ between $\mathbf{a}$ and $\mathbf{c}$ at shift τ can be written as

$$C_{\mathbf{a},\mathbf{c}}(\tau) = \begin{cases} \sum_{i=0}^{P-1} \omega_i \rho_i^* C_{\mathbf{s}}(\tau_1 + g_i - e_i), & \text{if } \tau = P\tau_1; \\ \sum_{i=0}^{P-1-\tau_0} \omega_i \rho_{i+\tau_0}^* C_{\mathbf{s}}(\tau_1 + g_{i+\tau_0} - e_i) + \sum_{i=P-\tau_0}^{P-1} \omega_i \rho_{i+\tau_0}^* C_{\mathbf{s}}(\tau_1 + 1 + g_{i+\tau_0} - e_i), & \\ \qquad \text{if } \tau = P\tau_1 + \tau_0 \text{ with } 0 < \tau_0 < P, & \end{cases} \tag{3}$$

where the index is reduced by modulo P.

Suppose that $\sum_{i=0}^{P-1} \omega_i \rho_i^* = 0$. From (3), if

$$\omega_0 \rho_{\tau_0}^* = \omega_1 \rho_{\tau_0+1}^* = \ldots = \omega_{P-1} \rho_{\tau_0-1}^* \tag{4}$$

holds for any $0 < \tau_0 < P$, then **a** and **c** are shift equivalent if and only if

$$e_0 - g_{\tau_0} = \ldots = e_{P-1-\tau_0} - g_{P-1} = e_{P-\tau_0} - g_0 - 1 = \ldots = e_{P-1} - g_{\tau_0-1} - 1.$$

Otherwise, **a** and **c** are shift distinct.

Example 1 *Put $P = 2$.*

- *Let $\omega_0 = 1, \omega_1 = 1, \rho_0 = 1$, and $\rho_1 = -1$. Then $\omega_0 \rho_0^* + \omega_1 \rho_1^* = 0$. One can check that (2) holds, but (4) doesn't hold.*
- *Let $\omega_0 = 1, \omega_1 = i, \rho_0 = 1$, and $\rho_1 = -i$, where $i^2 = -1$. Then $\omega_0 \rho_0^* + \omega_1 \rho_1^* = 0$. One can check that (2) doesn't hold, but (4) holds.*
- *Let $\omega_0 = 1, \omega_1 = e^{2\pi i/3}, \rho_0 = 1$, and $\rho_2 = -e^{2\pi i/3}$, where $i^2 = -1$. Then $\omega_0 \rho_0^* + \omega_1 \rho_1^* = 0$. One can check that both (2) and (4) do not hold.*

Definition 4 *With the notation as above, the difference matrix $D_{\mathbf{e},\mathbf{f}}$ between two shift sequences* **e** *and* **f** *is defined by*

$$\begin{pmatrix} e_0 - f_0 & e_1 - f_1 & \ldots & e_{P-2} - f_{P-2} & e_{P-1} - f_{P-1} \\ e_0 - f_1 & e_1 - f_2 & \ldots & e_{P-2} - f_{P-1} & e_{P-1} - f_0 - 1 \\ e_0 - f_2 & e_1 - f_3 & \ldots & e_{P-2} - f_0 - 1 & e_{P-1} - f_1 - 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ e_0 - f_{P-1} & e_1 - f_0 - 1 & \ldots & e_{P-2} - f_{P-3} - 1 & e_{P-1} - f_{P-2} - 1 \end{pmatrix}.$$

Besides, $\mathbf{e} \neq \mathbf{f}$ are called equivalent if at least one row of $D_{\mathbf{e},\mathbf{f}}$ is constant.

For convenience in the following sections, we fix some notation for $D_{\mathbf{e},\mathbf{f}}$. Suppose that

$$D_{\mathbf{e},\mathbf{f}} = \begin{pmatrix} \mathbf{d}_0 \\ \mathbf{d}_1 \\ \ldots \\ \mathbf{d}_{P-1} \end{pmatrix}.$$

We introduce the notation $\min_0(D_{\mathbf{e},\mathbf{f}})$, $\min_0^*(D_{\mathbf{e},\mathbf{f}})$, and $\text{Index}_0^*(D_{\mathbf{e},\mathbf{f}})$ which are defined by

$$\min_0(D_{\mathbf{e},\mathbf{f}}) = \min(\mathbf{d}_0),$$

$$\min_0^*(D_{\mathbf{e},\mathbf{f}}) = \min(\mathbf{d}_1, \mathbf{d}_2, ..., \mathbf{d}_{P-1}),$$

and

$$\text{Index}_0^*(D_{\mathbf{e},\mathbf{f}}) = \min_{1\le i<P}(\min(\mathbf{d}_i) = \min_0^*(D_{\mathbf{e},\mathbf{f}})).$$

3 New Construction of LCZ or ZCZ Sequence Sets

In this section, we propose a new construction of LCZ or ZCZ sequence sets.

Let

$$\begin{bmatrix} \omega_{0,0} & \omega_{0,1} & \cdots & \omega_{0,P-1} \\ \omega_{1,0} & \omega_{1,1} & \cdots & \omega_{1,P-1} \\ \cdots & \cdots & \cdots & \cdots \\ \omega_{P-1,0} & \omega_{P-1,1} & \cdots & \omega_{P-1,P-1} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega}_0 \\ \boldsymbol{\omega}_1 \\ \cdots \\ \boldsymbol{\omega}_{P-1} \end{bmatrix}$$

be a $P \times P$ complex Hadamard matrix [4], i.e., for any $0 \le i \ne j \le P-1$, $\sum_{k=0}^{P-1} \omega_{i,k}\omega_{j,k}^* = 0$, and for any $0 \le i, j \le P-1$, $|\omega_{i,j}| = 1$. Let $\mathbf{s} = \{s_i\}$ be a sequence with period N. Let $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_{M-1}\}$ be a set of inequivalent shift sequences, where $\mathbf{e}_i = \{e_{i,0}, e_{i,1}, ..., e_{i,P-1}\}$ with $0 \le e_{i,j} < N, 0 \le i < M, 0 \le j < P$. Let

$$\mathcal{S} = \{\ \mathbf{s}_{jM+i} = \mathbf{s}_{jM+i}(\mathbf{s}, \boldsymbol{\omega}_j, \mathbf{e}_i) \mid 0 \le i < M, 0 \le j < P\ \}.$$

Then $\mathcal{S}$ is a set of MP sequences with period NP.

Theorem 1 *With the notation as above, if the nontrivial autocorrelation of* $\mathbf{s}$ *is upper bounded by* δ*, then* $\mathcal{S}$ *is an LCZ sequence set with parameters* $(NP, MP, L, P\delta)$*, where*

$$L = \min\{\min_{\mathbf{e}_i\ne\mathbf{e}_j}\{P \cdot min_0(D_{\mathbf{e}_i,\mathbf{e}_j})\}, \min_{\mathbf{e}_i,\mathbf{e}_j}\{P \cdot min_0^*(D_{\mathbf{e}_i,\mathbf{e}_j}) + Index_0^*(D_{\mathbf{e}_i,\mathbf{e}_j})\}\}.$$

Proof. For any two sequences $\mathbf{a}$ and $\mathbf{b}$ in $\mathcal{S}$, we have

$$C_{\mathbf{a},\mathbf{b}}(-\tau) = C_{\mathbf{b},\mathbf{a}}^*(\tau).$$

Hence we only need to consider the case $\tau \ge 0$.

Let $\mathbf{a} = \mathbf{s}_{jM+i}(\mathbf{s}, \boldsymbol{\omega}_j, \mathbf{e}_i)$, and $\mathbf{b} = \mathbf{s}_{hM+t}(\mathbf{s}, \boldsymbol{\omega}_h, \mathbf{e}_t)$, where $0 \le i, t < M, 0 \le h, j < P$.

Case 1: computation of autocorrelation. For any $\tau = P\tau_1$, we have

$$C_{\mathbf{a}}(\tau) = P \cdot C_{\mathbf{s}}(\tau_1).$$

Hence $|C_{\mathbf{a}}(P\tau_1)| \leq P\delta$ for any $\tau_1 \neq 0$. For any $\tau = P\tau_1 + \tau_0$ with $0 < \tau_0 < P$, we have

$$\begin{aligned} C_{\mathbf{a}}(\tau) &= \sum_{k=0}^{P-1-\tau_0} \omega_{j,k}\omega_{j,k+\tau_0}^* C_{\mathbf{s}}(\tau_1 + e_{i,k+\tau_0} - e_{i,k}) \\ &\quad + \sum_{k=P-\tau_0}^{P-1} \omega_{j,k}\omega_{j,k+\tau_0}^* C_{\mathbf{s}}(\tau_1 + 1 + e_{i,k+\tau_0} - e_{i,k}). \end{aligned}$$

Hence we have $|C_{\mathbf{a}}(P\tau_1 + \tau_0)| \leq P\delta$ if $\tau_1 < \min_0^*(D_{\mathbf{e}_i,\mathbf{e}_i})$. If $\tau_1 = \min_0^*(D_{\mathbf{e}_i,\mathbf{e}_i})$, but $\tau_0 < \text{Index}_0^*(D_{\mathbf{e}_i,\mathbf{e}_i})$, we still have $|C_{\mathbf{a}}(P\tau_1 + \tau_0)| \leq P\delta$.

Case 2: computation of crosscorrelation.

1) $j = h, i \neq t$. In this case, for any $\tau = P\tau_1$, we have

$$C_{\mathbf{a},\mathbf{b}}(\tau) = \sum_{k=0}^{P-1} C_{\mathbf{s}}(\tau_1 + e_{t,k} - e_{i,k}).$$

Hence $|C_{\mathbf{a},\mathbf{b}}(P\tau_1)| \leq P\delta$ for any $\tau_1 < \min_0(D_{\mathbf{e}_i,\mathbf{e}_t})$. For any $\tau = P\tau_1 + \tau_0$ with $0 < \tau_0 < P$, we have

$$\begin{aligned} C_{\mathbf{a},\mathbf{b}}(\tau) &= \sum_{k=0}^{P-1-\tau_0} \omega_{j,k}\omega_{j,k+\tau_0}^* C_{\mathbf{s}}(\tau_1 + e_{t,k+\tau_0} - e_{i,k}) \\ &\quad + \sum_{k=P-\tau_0}^{P-1} \omega_{j,k}\omega_{j,k+\tau_0}^* C_{\mathbf{s}}(\tau_1 + 1 + e_{t,k+\tau_0} - e_{i,k}). \end{aligned}$$

Hence $|C_{\mathbf{a},\mathbf{b}}(P\tau_1 + \tau_0)| \leq P\delta$ if $\tau_1 < \min_0^*(D_{\mathbf{e}_i,\mathbf{e}_t})$. If $\tau_1 = \min_0^*(D_{\mathbf{e}_i,\mathbf{e}_t})$, but $\tau_0 < \text{Index}_0^*(D_{\mathbf{e}_i,\mathbf{e}_t})$, we still have $|C_{\mathbf{a},\mathbf{b}}(P\tau_1 + \tau_0)| \leq P\delta$.

2) $j \neq h$. In this case, for any $\tau = P\tau_1$, we have

$$C_{\mathbf{a},\mathbf{b}}(\tau) = \sum_{k=0}^{P-1} \omega_{j,k}\omega_{h,k}^* C_{\mathbf{s}}(\tau_1 + e_{t,k} - e_{i,k}).$$

If $i = t$, then for any τ_1, $C_{\mathbf{a},\mathbf{b}}(\tau) = C_{\mathbf{s}}(\tau_1)\sum_{k=0}^{P-1} \omega_{j,k}\omega_{h,k}^* = 0$. If $i \neq t$, then $|C_{\mathbf{a},\mathbf{b}}(\tau)| \leq P\delta$ for any $\tau_1 < \min_0(D_{\mathbf{e}_i,\mathbf{e}_t})$.

For any $\tau = P\tau_1 + \tau_0$ with $0 < \tau_0 < P$, we have

$$\begin{aligned} C_{\mathbf{a},\mathbf{b}}(\tau) &= \sum_{k=0}^{P-1-\tau_0} \omega_{j,k}\omega_{h,k+\tau_0}^* C_{\mathbf{s}}(\tau_1 + e_{t,k+\tau_0} - e_{i,k}) \\ &\quad + \sum_{k=P-\tau_0}^{P-1} \omega_{j,k}\omega_{h,k+\tau_0}^* C_{\mathbf{s}}(\tau_1 + 1 + e_{t,k+\tau_0} - e_{i,k}). \end{aligned}$$

Hence $|C_{\mathbf{a},\mathbf{b}}(P\tau_1+\tau_0)| \le P\delta$ if $\tau_1 < \min_0^*(D_{\mathbf{e}_i,\mathbf{e}_t})$. If $\tau_1 = \min_0^*(D_{\mathbf{e}_i,\mathbf{e}_t})$, but $\tau_0 < \mathrm{Index}_0^*(D_{\mathbf{e}_i,\mathbf{e}_t})$, we still have $|C_{\mathbf{a},\mathbf{b}}(P\tau_1+\tau_0)| \le P\delta$.

Finally, by the analysis above, $\mathcal{S}$ is an LCZ sequence set with parameters $(NP, MP, L, P\delta)$. □

4 Constructions of Shift Sequences

In this section, we present the set of shift sequences $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_{M-1}\}$ used in the construction of $\mathcal{S}$ in Section 3.

The main results of this section are as follows.

Case 1: $P|L$

Let

$$M = \left\lfloor \frac{N-P-\sigma}{L} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ is the floor function, and

$$\sigma = \begin{cases} 0, & \text{if } P=2 \text{ or } L|N-1; \\ 1, & \text{otherwise.} \end{cases} \tag{5}$$

For any $0 \le i < M$, if $L|N-1$ or $P > 2$, let

$$\mathbf{e}_i = \left\{ \frac{iL}{P}, \frac{ML}{P} + \frac{(M-1-i)L}{P} + 2, \frac{2ML}{P} + \frac{iL}{P} + 3, ..., \frac{(P-1)ML}{P} + \frac{iL}{P} + P \right\}; \tag{6}$$

if $L \not| N-1$ and $P = 2$, let

$$\mathbf{e}_i = \left\{ \frac{iL}{P}, \frac{ML}{P} + \frac{(M-1-i)L}{P} + 1 \right\}. \tag{7}$$

Now we show an example of shift sequences defined by (6).

Example 2 *Let $P = 4$, $L = 8$, and $N = 30$. Then $M = 3$, and the shift sequence $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\}$ is defined as follows: $\mathbf{e}_0 = \{0, 12, 15, 22\}, \mathbf{e}_1 = \{2, 10, 17, 24\}$, and $\mathbf{e}_2 = \{4, 8, 19, 26\}$.*

	0	2	4	8	10	12	15	17	19	22	24	26
$\mathbf{e}_0$	$e_{0,0}$					$e_{0,1}$	$e_{0,2}$			$e_{0,3}$		
$\mathbf{e}_1$		$e_{1,0}$			$e_{1,1}$			$e_{1,2}$			$e_{1,3}$	
$\mathbf{e}_2$			$e_{2,0}$	$e_{2,1}$					$e_{2,2}$			$e_{2,3}$

Theorem 2 *Suppose that $P < L$ and $P|L$. Let $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_{M-1}\}$ be the set of shift sequences defined by (6) or (7). If the nontrivial autocorrelation of $\mathbf{s}$ is upper bounded by δ, then $\mathcal{S}$ constructed in Section 3 is an LCZ sequence set with parameters $(NP, MP, L, P\delta)$. In particular, if $\delta = 0$, then $\mathcal{S}$ is a ZCZ sequence set with parameters (NP, MP, L).*

Case 2: P Even, and $L \equiv P/2 \pmod{P}$

For simplicity, we denote $\lfloor L/P \rfloor$ by q. Let

$$M = \left\lfloor \frac{N - 3P/2 + 1 - \sigma}{L} \right\rfloor,$$

where

$$\sigma = \begin{cases} 0, & \text{if } P = 2 \text{ or } 4; \\ 1, & \text{otherwise.} \end{cases} \tag{8}$$

For $0 \le j \le P/2$, define δ_j by

$$\delta_j = \begin{cases} 0, & \text{if } j = 0, \\ P/2, & \text{if } j = P/2 \text{ and } P > 4, \\ j - 1, & \text{otherwise .} \end{cases}$$

For any $0 \le i < M$, if $L \nmid N - 2$ or $P > 2$, let

$$e_{i,j} = \begin{cases} j(M(2q+1)+2) + \delta_j + (2q+1)i/2, & \text{if } i \text{ is even ,} \\ (j+1)(M(2q+1)+2) + \delta_{j+1} - 1 - (2q+1)(i+1)/2, & \text{if } i \text{ is odd ,} \end{cases} \tag{9}$$

and

$$e_{i,j+P/2} = \begin{cases} (j+1)(M(2q+1)+2) + \delta_{j+1} - 1 - q - (2q+1)i/2, & \text{if } i \text{ is even ,} \\ j(M(2q+1)+2) + \delta_j + (2q+1)(i-1)/2 + q + 1, & \text{if } i \text{ is odd ,} \end{cases} \tag{10}$$

where $j = 0, 1, ..., P/2 - 1$.

If $L | N - 2$ and $P = 2$, let

$$M = \left\lfloor \frac{N-1}{L} \right\rfloor.$$

For any $0 \le i < M$, let

$$e_{i,0} = \begin{cases} (2q+1)i/2, & \text{if } i \text{ is even ,} \\ N - (2q+1)(i+1)/2, & \text{if } i \text{ is odd ,} \end{cases} \tag{11}$$

and

$$e_{i,1} = \begin{cases} N - q - (2q+1)i/2, & \text{if } i \text{ is even ,} \\ (2q+1)(i-1)/2 + q + 1, & \text{if } i \text{ is odd .} \end{cases} \tag{12}$$

Now we show an example of shift sequences defined by (9) and (10).

Example 3 *Let $P = 4$, $L = 10$, and $N = 50$. Then $M = 4$, and the shift sequence $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is defined as follows: $\mathbf{e}_0 = \{0, 22, 19, 42\}, \mathbf{e}_1 = \{16, 39, 3, 25\}, \mathbf{e}_2 = \{5, 27, 14, 37\}$, and $\mathbf{e}_3 = \{11, 34, 8, 30\}$.*

	0	3	5	8	11	14	16	19	22	25	27	30	34	37	39	42
$\mathbf{e}_0$	$e_{0,0}$							$e_{0,2}$	$e_{0,1}$							$e_{0,3}$
$\mathbf{e}_1$		$e_{1,2}$					$e_{1,0}$			$e_{1,3}$					$e_{1,1}$	
$\mathbf{e}_2$			$e_{2,0}$			$e_{2,2}$					$e_{2,1}$			$e_{2,3}$		
$\mathbf{e}_3$				$e_{3,2}$	$e_{3,0}$							$e_{3,3}$	$e_{3,1}$			

Theorem 3 *Suppose that P is even, and $L \equiv P/2 \pmod{P}$. Let $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_{M-1}\}$ be the set of shift sequences defined by (9) and (10), or (11) and (12). If the nontrivial autocorrelation of $\mathbf{s}$ is upper bounded by δ, then $\mathcal{S}$ constructed in Section 3 is an LCZ sequence set with parameters $(NP, MP, L, P\delta)$. In particular, if $\delta = 0$, then $\mathcal{S}$ is a ZCZ sequence set with parameters (NP, MP, L).*

In order to prove Theorems 2 and 3, we need a number of lemmas which are provided in the following subsections.

4.1 The Case of $P|L$

Lemma 1 *For any $0 \leq i < M$, let*

$$D_{\mathbf{e}_i, \mathbf{e}_i} = \begin{pmatrix} \mathbf{0} \\ \mathbf{d}_1 \\ ... \\ \mathbf{d}_{P-1} \end{pmatrix}.$$

If $P < L$, then $\mathbf{d}_j$ is not constant for any $1 \leq j < P$.

Proof. The proof can be divided into 3 cases.

1) $P = 2$. In this case,

$$\mathbf{d}_1 = \begin{cases} (N - ML + iL + L/2 - 2, ML - L/2 - iL + 1), & \text{if } L | N - 1; \\ (N - ML + iL + L/2 - 1, ML - L/2 - iL), & \text{if } L \not| N - 1. \end{cases}$$

Because $L > P \geq 2$, $\mathbf{d}_1$ is not constant.

2) $P = 3$. In this case, $\mathbf{d}_1 = (N - 2ML/3 + 2iL/3 + L/3 - 2, N - 2iL/3 - L/3 - 1, 2ML/3 + 2)$, and $\mathbf{d}_2 = (N - 2ML/3 - 3, 2ML/3 - 2iL/3 - L/3 + 1, 2iL/3 + L/3)$. Hence both $\mathbf{d}_1$ and $\mathbf{d}_2$ are not constant.

3) $P > 3$. Suppose that $\mathbf{d}_j$ is constant for some $1 \leq j < P$. If $j = 1$, then we have

$$N + \frac{iL}{P} - \frac{ML}{P} - \frac{(M-1-i)L}{P} - 2 = N + \frac{ML}{P} + \frac{(M-1-i)L}{P} + 2 - \frac{2ML}{P} - \frac{iL}{P} - 3.$$

It follows that

$$\frac{2iL}{P} - \frac{2ML}{P} + \frac{L}{P} - 1 = -\frac{L}{P} - \frac{2iL}{P}.$$

Hence $(L/P)|1$. It is a contradiction.

If $1 < j < P-1$, then we have

$$\begin{aligned} & N + \frac{iL}{P} - \left(\frac{jML}{P} + \frac{iL}{P} + j + 1\right) \\ & = N + \frac{ML}{P} + \frac{(M-1-i)L}{P} + 2 - \left(\frac{(j+1)ML}{P} + \frac{iL}{P} + j + 2\right). \end{aligned}$$

It follows that $(M-1-2i)L/P = -1$ which is a contradiction.

If $j = P-1$, then we have

$$\frac{ML}{P} + \frac{(M-1-i)L}{P} + 2 - \frac{iL}{P} - 1 = \frac{2ML}{P} + \frac{iL}{P} + 3 - \left(\frac{ML}{P} + \frac{(M-1-i)L}{P} + 2\right) - 1.$$

It follows that

$$\frac{(2M-2i-1)L}{P} + 1 = \frac{2iL}{P} + \frac{L}{P}.$$

Hence $(L/P)|1$. We get a contradiction again. □

Lemma 2 *For any $0 \le i \ne j < M$, if $P < L$, then $\mathbf{e}_i$ and $\mathbf{e}_j$ are inequivalent.*

Proof. We can assume that $i > j$ because the proof for the case $i < j$ is similar. Let

$$D_{\mathbf{e}_i,\mathbf{e}_j} = \begin{pmatrix} \mathbf{d}_0 \\ \mathbf{d}_1 \\ \dots \\ \mathbf{d}_{P-1} \end{pmatrix}.$$

We divide the proof into three cases.

1) $P = 2$.

If $L|N-1$, then $\mathbf{d}_0 = ((i-j)L/2, N-(i-j)L/2)$, and $\mathbf{d}_1 = (N - ML + (i+j)L/2 + L/2 - 2, ML - L/2 - (i+j)L/2 + 1)$. If $\mathbf{d}_0$ is constant, then $(L/2)|N$ which is a contradiction. If $\mathbf{d}_1$ is constant, then $L|2$ which is also a contradiction.

If $L \not| N-1$, then $\mathbf{d}_0 = ((i-j)L/2, N-(i-j)L/2)$, and $\mathbf{d}_1 = (N - ML + (i+j)L/2 + L/2 - 1, ML - L/2 - (i+j)L/2)$. Because

$$N - (i-j)L/2 > ML - (i-j)L/2 > ML/2 > (i-j)L/2,$$

$\mathbf{d}_0$ is not constant. If $\mathbf{d}_1$ is constant, then $L|N-1$ which is also a contradiction.

2) $P=3$. In this case, $\mathbf{d}_0=((i-j)L/3, N-(i-j)L/3, (i-j)L/3)$, $\mathbf{d}_1=(N-2ML/3+(i+j)L/3+L/3-2, N-(i+j)L/3-L/3-1, (i-j)L/3+2ML/3+2)$, and $\mathbf{d}_2=(N-2ML/3+(i-j)L/3-3, 2ML/3-(i+j)L/3-L/3+1, (i+j)L/3+L/3)$. Because

$$N-(i-j)L/3 > ML-(i-j)L/3 > 2ML/3 > (i-j)L/3,$$

$\mathbf{d}_0$ is not constant. If $\mathbf{d}_1$ or $\mathbf{d}_2$ is constant, then $(L/3)|1$ which is a contradiction.

3) $P>3$. In this case, $\mathbf{d}_0=((i-j)L/P, N-(i-j)L/P, (i-j)L/P, ..., (i-j)L/P)$. Because

$$N-(i-j)L/P > ML-(i-j)L/P > (P-1)ML/P > (i-j)L/P,$$

$\mathbf{d}_0$ is not constant. Suppose that $\mathbf{d}_k$ is constant for some $1\le k<P$.

If $k=1$, then we have

$$N+\frac{iL}{P}-\frac{ML}{P}-\frac{(M-1-j)L}{P}-2=N+\frac{ML}{P}+\frac{(M-1-i)L}{P}+2-\frac{2ML}{P}-\frac{jL}{P}-3.$$

Hence

$$\frac{(i+j)L}{P}-\frac{2ML}{P}+\frac{L}{P}-1=-\frac{L}{P}-\frac{(i+j)L}{P}.$$

It is a contradiction.

If $1<k<P-1$, then we have

$$\begin{aligned} &N+\frac{iL}{P}-\left(\frac{kML}{P}+\frac{jL}{P}+k+1\right) \\ &\quad =N+\frac{ML}{P}+\frac{(M-1-i)L}{P}+2-\left(\frac{(k+1)ML}{P}+\frac{jL}{P}+k+2\right). \end{aligned}$$

It follows that $(M-1-2i)L/P=-1$ which is a contradiction.

If $k=P-1$, then we have

$$\frac{ML}{P}+\frac{(M-1-i)L}{P}+2-\frac{jL}{P}-1=\frac{2ML}{P}+\frac{iL}{P}+3-\left(\frac{ML}{P}+\frac{(M-1-j)L}{P}+2\right)-1.$$

It follows that

$$\frac{(2M-i-j-1)L}{P}+1=\frac{(i+j)L}{P}+\frac{L}{P}.$$

We get a contradiction again. □

Lemma 3 *For any $0\le i\ne j<M$, $min_0(D_{\mathbf{e}_i,\mathbf{e}_j})\ge L/P$, and for any $0\le i,j<M$, $min_0^*(D_{\mathbf{e}_i,\mathbf{e}_j})\ge L/P$.*

Proof. If $i > j$, then

$$\begin{aligned}
&\min_0(D_{\mathbf{e}_i,\mathbf{e}_j}) \\
&= \min((i-j)L/P, N-(i-j)L/P, (i-j)L/P, ..., (i-j)L/P) \\
&\geq L/P.
\end{aligned}$$

If $i < j$, then

$$\begin{aligned}
&\min_0(D_{\mathbf{e}_i,\mathbf{e}_j}) \\
&= \min(N+(i-j)L/P, (j-i)L/P, N+(i-j)L/P, ..., N+(i-j)L/P) \\
&\geq L/P.
\end{aligned}$$

Hence $\min_0(D_{\mathbf{e}_i,\mathbf{e}_j}) \geq L/P$.

If $L \not| N-1$ and $P = 2$, then

$$\begin{aligned}
e_{i,0} - e_{j,1} &= N + \frac{iL}{P} - \left(\frac{ML}{P} + \frac{(M-1-j)L}{P} + 1\right) \\
&\geq N - \frac{ML}{P} - \frac{(M-1)L}{P} - 1 > \frac{L}{P},
\end{aligned}$$

and

$$\begin{aligned}
e_{i,1} - e_{j,0} - 1 &= \frac{ML}{P} + \frac{(M-1-i)L}{P} + 1 - \frac{jL}{P} - 1 \\
&\geq \frac{ML}{P} - \frac{(M-1)L}{P} = \frac{L}{P}.
\end{aligned}$$

Hence $\min_0^*(D_{\mathbf{e}_i,\mathbf{e}_j}) \geq L/P$ in this case.

Suppose that $L|N-1$ or $P > 2$. For any $0 < k \leq P-1$,

$$\begin{aligned}
e_{i,0} - e_{j,k} &\geq N + \frac{iL}{P} - \left(\frac{kML}{P} + \frac{(M-1)L}{P} + k + 1\right) \\
&\geq N - \left(ML - \frac{L}{P} + P\right) > \frac{L}{P},
\end{aligned}$$

and

$$e_{i,k} - e_{j,0} - 1 \geq \frac{kML}{P} + k + 1 - \frac{jL}{P} - 1 \geq \frac{L}{P} + 1.$$

For any $1 \leq k_1 < k_2 \leq P-1$,

$$\begin{aligned}
e_{i,k_1} - e_{j,k_2} &> N - \left(\frac{k_2ML}{P} + \frac{(M-1)L}{P} + k_2 + 1\right) \\
&\geq N - \left(ML - \frac{L}{P} + P\right) > \frac{L}{P},
\end{aligned}$$

and

$$e_{i,k_2} - e_{j,k_1} - 1 \geq \frac{k_2 ML}{P} + k_2 + 1 - \left(\frac{k_1 ML}{P} + \frac{(M-1)L}{P} + k_1 + 1\right) - 1 \geq \frac{L}{P}.$$

Hence $\min_0^*(D_{\mathbf{e}_i,\mathbf{e}_j}) \geq L/P$. □

Proof of Theorem 2: By Theorem 1, Lemmas 1, 2, and 3, the result follows. □

4.2 The Case of P Even, and $L \equiv P/2 \pmod P$

Lemma 4 *If $P > 2$, then $N - (M(2q+1)+2)P/2 \geq P/2 - 1 + \sigma$.*

Proof. Because $(M(2q+1)+2)P/2 = ML + P$, we have

$$N - (M(2q+1)+2)P/2 \geq ML + 3P/2 - 1 + \sigma - (ML + P) = P/2 - 1 + \sigma.$$

□

Lemma 5 *For any $0 \leq i < M$, let*

$$D_{\mathbf{e}_i,\mathbf{e}_i} = \begin{pmatrix} \mathbf{0} \\ \mathbf{d}_1 \\ \cdots \\ \mathbf{d}_{P-1} \end{pmatrix}.$$

If $L > P$, then $\mathbf{d}_j$ is not constant for any $1 \leq j < P$.

Proof. The proof can be divided into 2 steps.

1) $P = 2$.

If $L \not| N-2$ and i is even, then $\mathbf{d}_1 = (N + (2q+1)i - M(2q+1) - 1 + q, M(2q+1) - q - (2q+1)i)$. Hence $\mathbf{d}_1$ is not constant.

If $L \not| N-2$ and i is odd, then $\mathbf{d}_1 = (M(2q+1) - (2q+1)(i-1)/2 - 3q - 1, N + (2q+1)(i-1)/2 - M(2q+1) + 3q)$. Hence $\mathbf{d}_1$ is not constant.

If $L|N-2$ and i is even, then $\mathbf{d}_1 = ((2q+1)i + q, N - q - (2q+1)i - 1)$. If $\mathbf{d}_1$ is constant, then $L|N$. It follows that $L|2$. Because $L > P \geq 2$, $\mathbf{d}_1$ is not constant.

If $L|N-2$ and i is odd, then $\mathbf{d}_1 = (N - (2q+1)(i-1)/2 - 3q - 2, (2q+1)(i-1) + 3q + 1)$. If $\mathbf{d}_1$ is constant, then $L|N$. It follows that $L|2$. Because $L > P \geq 2$, $\mathbf{d}_1$ is not constant.

2) $P > 2$. Suppose that $\mathbf{d}_k$ is constant for some $1 \leq k < P$.

- i is even

  If $1 \leq k < P/2$, then we have $e_{i,0} - e_{i,k} = e_{i,P/2} - e_{i,P/2+k}$. Hence

$$k(M(2q+1)+2) + \delta_k = k(M(2q+1)+2) + \delta_{k+1}.$$

It is a contradiction.

If $k = P/2$, then we have $e_{i,0} - e_{i,P/2} = e_{i,P/2} - e_{i,0} - 1$. Because

$$e_{i,0} - e_{i,P/2} = N - (M(2q+1)+2) + 1 + q + (2q+1)i > M(2q+1) + 2,$$

and

$$e_{i,P/2} - e_{i,0} - 1 = (M(2q+1)+2) - 1 - q - (2q+1)i < M(2q+1) + 2,$$

we get a contradiction.

If $P/2 < k < P$, then we have $e_{i,P-1} - e_{i,k-1} - 1 = e_{i,P/2-1} - e_{i,k-P/2-1} - 1$. Hence

$$(P-k)(M(2q+1)+2) + \delta_{P/2} - \delta_{k-P/2} = (P-k)(M(2q+1)+2) + \delta_{P/2-1} - \delta_{k-P/2-1}.$$

If $k - P/2 - 1 = 0$, then $\delta_{P/2} - \delta_{k-P/2} = \delta_{P/2} > \delta_{P/2-1} = \delta_{P/2-1} - \delta_{k-P/2-1}$; If $k - P/2 - 1 \geq 1$, then $\delta_{P/2} - \delta_{k-P/2} = P - k + 1 > P - k = \delta_{P/2-1} - \delta_{k-P/2-1}$. It is a contradiction.

- i is odd

  In this case, the proof is similar to that of i even. So we omit it.

□

Lemma 6 *For any $0 \leq i \neq j < M$, if $L > P$, then $\mathbf{e}_i$ and $\mathbf{e}_j$ are inequivalent.*

Proof. The proof can be divided into 4 cases: 1) i even, j even; 2) i even, j odd; 3) i odd, j even; 4) i odd, j odd. We only need to prove the first two cases because the proof for the other two cases is similar.

Let

$$D_{\mathbf{e}_i,\mathbf{e}_j} = \begin{pmatrix} \mathbf{d}_0 \\ \mathbf{d}_1 \\ \dots \\ \mathbf{d}_{P-1} \end{pmatrix}.$$

Firstly, we prove the case of i even and j even. We can assume that $i > j$ because the proof for the case $i < j$ is similar.

1) $P = 2$.

If $L \nmid N-2$, then $\mathbf{d}_0 = ((2q+1)(i-j)/2, N-(2q+1)(i-j)/2)$, and $\mathbf{d}_1 = (N+(2q+1)i-M(2q+1)-1+q, M(2q+1)-q-(2q+1)i)$. Hence $\mathbf{d}_0$ and $\mathbf{d}_1$ are not constant.

If $L|N-2$, then $\mathbf{d}_0 = ((2q+1)(i-j)/2, N-(2q+1)(i-j)/2)$, and $\mathbf{d}_1 = ((2q+1)i+q, N-q-(2q+1)i-1)$. Hence $\mathbf{d}_0$ is not constant. Because $L > 2$, $\mathbf{d}_1$ is not constant.

2) $P > 2$.

Suppose that $\mathbf{d}_k$ is constant for some $0 \le k < P$.

If $k = 0$, then $e_{i,0} - e_{j,0} = e_{i,P/2} - e_{j,P/2}$. It follows that $(2q+1)(i-j)/2 = N - (2q+1)(i-j)/2$. It is a contradiction.

If $1 \le k < P/2$, then $e_{i,0} - e_{j,k} = e_{i,P/2} - e_{j,P/2+k}$. It follows that

$$N - k(M(2q+1)+2) - \delta_k + (2q+1)(i-j)/2 = N - k(M(2q+1)+2) - \delta_{k+1} - (2q+1)(i-j)/2.$$

It is a contradiction.

If $k = P/2$, then $e_{i,0} - e_{j,P/2} = e_{i,P/2} - e_{j,0} - 1$. It follows that

$$N - M(2q+1) + (2q+1)(i+j)/2 + q - 1 = M(2q+1) - (2q+1)(i+j)/2 - q.$$

Because $N - M(2q+1) > M(2q+1)$, it is a contradiction.

If $P/2 < k < P$, then $e_{i,P-1} - e_{j,k-1} - 1 = e_{i,P/2-1} - e_{j,k-P/2-1} - 1$. We have

$$\begin{aligned} &(P-k)(M(2q+1)+2) + \delta_{P/2} - \delta_{k-P/2} - (2q+1)(i-j)/2 \\ = \quad &(P-k)(M(2q+1)+2) + \delta_{P/2-1} - \delta_{k-P/2-1} + (2q+1)(i-j)/2. \end{aligned}$$

It follows that

$$\delta_{P/2} - \delta_{k-P/2} - (\delta_{P/2-1} - \delta_{k-P/2-1}) = (2q+1)(i-j).$$

We get a contradiction.

Now we prove the case of i even and j odd.

1) $P = 2$.

If $L \not| N-2$, then $\mathbf{d}_0 = (N - M(2q+1) - 1 + (2q+1)(i+j+1)/2, M(2q+1) - (2q+1)(i+j-1)/2 - 2q)$, and

$$\mathbf{d}_1 = \begin{cases} ((2q+1)(i-j+1)/2 - q - 1, N - (2q+1)(i-j-1)/2 - q - 1), & \text{if } i > j-1; \\ (N + (2q+1)(i-j+1)/2 - q - 1, -(2q+1)(i-j-1)/2 - q - 1), & \text{otherwise.} \end{cases}$$

Hence $\mathbf{d}_0$ and $\mathbf{d}_1$ are not constant.

If $L | N-2$, then $\mathbf{d}_0 = ((2q+1)(i+j+1)/2, N - 2q - 1 - (2q+1)(i+j-1)/2)$, and

$$\mathbf{d}_1 = \begin{cases} ((2q+1)(i-j+1)/2 - q - 1, N - (2q+1)(i-j-1)/2 - q - 1), & \text{if } i > j-1; \\ (N + (2q+1)(i-j+1)/2 - q - 1, -(2q+1)(i-j-1)/2 - q - 1), & \text{otherwise.} \end{cases}$$

Hence $\mathbf{d}_0$ and $\mathbf{d}_1$ are not constant.

2) $P > 2$.

Suppose that $\mathbf{d}_k$ is constant for some $0 \le k < P$.

If $k=0$, then $e_{i,0}-e_{j,0}=e_{i,P/2}-e_{j,P/2}$. It follows that

$$N-M(2q+1)-1+(2q+1)(i+j+1)/2=M(2q+1)-2q-(2q+1)(i+j-1)/2.$$

It is a contradiction.

If $1\le k<P/2$, then $e_{i,0}-e_{j,k}=e_{i,P/2}-e_{j,P/2+k}$. We have

$$\begin{aligned} & N-(k+1)(M(2q+1)+2)-\delta_{k+1}+1+(2q+1)(i+j+1)/2 \\ = \quad & N-(k-1)(M(2q+1)+2)-2-2q-\delta_k-(2q+1)(i+j-1)/2. \end{aligned}$$

It follows that

$$(2q+1)(i+j)+1=2(M(2q+1)+2)-2-2q+\delta_{k+1}-\delta_k.$$

It is a contradiction.

If $k=P/2$, then $e_{i,0}-e_{j,P/2}=e_{i,P/2}-e_{j,0}-1$. It follows that

$$(2q+1)(i-j+1)/2-q-1=N-(2q+1)(i-j-1)/2-q-1$$

if $i>j-1$, and

$$N+(2q+1)(i-j+1)/2-q-1=-(2q+1)(i-j-1)/2-q-1$$

if $i\le j-1$. We get a contradiction.

If $P/2<k<P$, then $e_{i,P-1}-e_{j,k-1}-1=e_{i,P/2-1}-e_{j,k-P/2-1}-1$. We have

$$\begin{aligned} & (P-k+1)(M(2q+1)+2)+\delta_{P/2}-2-2q-\delta_{k-1-P/2}-(2q+1)(i+j-1)/2 \\ = \quad & (P-1-k)(M(2q+1)+2)+\delta_{P/2-1}-\delta_{k-P/2}-2+(2q+1)(i+j+1)/2. \end{aligned}$$

It follows that

$$2(M(2q+1)+2)+\delta_{P/2}-2q-\delta_{k-1-P/2}=\delta_{P/2-1}-\delta_{k-P/2}+(2q+1)(i+j).$$

We get a contradiction again. □

Lemma 7 *For any* $0\le i\ne j<M$, $min_0(D_{\mathbf{e}_i,\mathbf{e}_j})\ge 2q+1$.

Proof. Similar to the proof of Lemma 6, we only need to prove 2 cases: 1) i even, j even; 2) i even, j odd.

Let $\mathbf{d}_0$ be the first row of $D_{\mathbf{e}_i,\mathbf{e}_j}$. Firstly, we prove the case of i even and j even. We can assume that $i>j$ because the proof for the case $i<j$ is similar.

1) $P=2$. In this case, $\mathbf{d}_0=((2q+1)(i-j)/2, N-(2q+1)(i-j)/2)$. Hence $\min_0(D_{\mathbf{e}_i,\mathbf{e}_j})\ge 2q+1$.

2) $P > 2$. In this case, $e_{i,k} - e_{j,k} = (2q+1)(i-j)/2$, and $e_{i,k+P/2} - e_{j,k+P/2} = N - (2q+1)(i-j)/2$ for any $0 \le k < P/2$. Hence $\min_0(D_{\mathbf{e}_i,\mathbf{e}_j}) \ge 2q+1$.

Now we prove the case of i even and j odd.

1) $P = 2$.

If $L \not| N-2$, then $\mathbf{d}_0 = (N - M(2q+1) - 1 + (2q+1)(i+j+1)/2, M(2q+1) - (2q+1)(i+j-1)/2 - 2q)$. Hence $\min_0(D_{\mathbf{e}_i,\mathbf{e}_j}) > 2q+1$.

If $L | N-2$, then $\mathbf{d}_0 = ((2q+1)(i+j+1)/2, N - 2q - 1 - (2q+1)(i+j-1)/2)$. Hence $\min_0(D_{\mathbf{e}_i,\mathbf{e}_j}) \ge 2q+1$.

2) $P > 2$. In this case, $e_{i,k} - e_{j,k} = N - M(2q+1) - 1 + \delta_k - \delta_{k+1} + (2q+1)(i+j+1)/2$, and $e_{i,k+P/2} - e_{j,k+P/2} = M(2q+1) + \delta_{k+1} - \delta_k - 2q - (2q+1)(i+j-1)/2$ for any $0 \le k < P/2$. Hence $\min_0(D_{\mathbf{e}_i,\mathbf{e}_j}) > 2q+1$. □

Lemma 8 *For any $0 \le i, j < M$, $min_0^*(D_{\mathbf{e}_i,\mathbf{e}_j}) \ge q$. In particular, if $min_0^*(D_{\mathbf{e}_i,\mathbf{e}_j}) = q$, then $Index_0^*(D_{\mathbf{e}_i,\mathbf{e}_j}) = P/2$.*

Proof. Similar to the proof of Lemma 6, we only need to prove 2 cases: 1) i even, j even; 2) i even, j odd.

Let

$$D_{\mathbf{e}_i,\mathbf{e}_j} = \begin{pmatrix} \mathbf{d}_0 \\ \mathbf{d}_1 \\ \cdots \\ \mathbf{d}_{P-1} \end{pmatrix}.$$

Firstly, we prove the case of i even and j even. We can assume that $i \ge j$ because the proof for the case $i < j$ is similar.

We omit the proof for $P = 2$ since it is straightforward.

For $P > 2$, we consider the differences of $e_{i,k_1} - e_{j,k_2}$ and $e_{i,k_2} - e_{j,k_1} - 1$ for the following 5 cases of (k_1, k_2): 1) $0 \le k_1 < k_2 < P/2$; 2) $0 \le k_1 < P/2 < k_2 < P$ with $k_1 < k_2 - P/2$; 3) $0 \le k_1 < P/2 < k_2 < P$ with $k_1 > k_2 - P/2$; 4) $P/2 < k_1 < k_2 < P$; and 5) $k_2 = k_1 + P/2$.

1) For any $0 \le k_1 < k_2 < P/2$, we have

$$e_{i,k_1} - e_{j,k_2} = N - (k_2 - k_1)(M(2q+1) + 2) + \delta_{k_1} - \delta_{k_2} + (2q+1)(i-j)/2.$$

By Lemma 4, it follows that

$$e_{i,k_1} - e_{j,k_2} \ge M(2q+1) + 2 + P/2 - 1 + \delta_{k_1} - \delta_{k_2} + (2q+1)(i-j)/2 > q.$$

Moreover, we have

$$e_{i,k_2} - e_{j,k_1} - 1 = (k_2 - k_1)(M(2q+1) + 2) - \delta_{k_1} + \delta_{k_2} + (2q+1)(j-i)/2 - 1 > q.$$

2) For any $0 \le k_1 < P/2 < k_2 < P$ with $k_1 < k_2 - P/2$, we have

$$e_{i,k_1} - e_{j,k_2} = N + (k_1 - k_2 + P/2 - 1)(M(2q+1)+2) + \delta_{k_1} - \delta_{k_2-P/2+1} + 1 + q + (2q+1)(i+j)/2.$$

By Lemma 4, it follows that

$$e_{i,k_1} - e_{j,k_2} \ge P/2 - 1 + \sigma + \delta_{k_1} - \delta_{k_2-P/2+1} + 1 + q + (2q+1)(i+j)/2 > q.$$

Moreover, we have

$$e_{i,k_2} - e_{j,k_1} - 1 = (k_2 - P/2 + 1 - k_1)(M(2q+1)+2) - \delta_{k_1} + \delta_{k_2-P/2+1} - 2 - q - (2q+1)(i+j)/2 > q.$$

3) For any $0 \le k_1 < P/2 < k_2 < P$ with $k_1 > k_2 - P/2$, we have

$$e_{i,k_1} - e_{j,k_2} = (k_1 - k_2 + P/2 - 1)(M(2q+1)+2) + \delta_{k_1} - \delta_{k_2-P/2+1} + 1 + q + (2q+1)(i+j)/2 > q.$$

Moreover, we have

$$\begin{aligned} e_{i,k_2} - e_{j,k_1} - 1 &= N + (k_2 - P/2 + 1 - k_1)(M(2q+1)+2) - \delta_{k_1} + \delta_{k_2-P/2+1} \\ &\quad -2 - q - (2q+1)(i+j)/2 \\ &\ge 2(M(2q+1)+2) + P/2 - 1 - \delta_{k_1} + \delta_{k_2-P/2+1} - 2 - q - (2q+1)(i+j)/2 \\ &> q. \end{aligned}$$

4) For any $P/2 < k_1 < k_2 < P$, we have

$$e_{i,k_1} - e_{j,k_2} = N - (k_2 - k_1)(M(2q+1)+2) + \delta_{k_1-P/2} - \delta_{k_2-P/2} + (2q+1)(i-j)/2.$$

By Lemma 4, it follows that

$$e_{i,k_1} - e_{j,k_2} \ge M(2q+1) + 2 + P/2 - 1 + \delta_{k_1-P/2} - \delta_{k_2-P/2} + (2q+1)(i-j)/2 > q.$$

Moreover, we have

$$e_{i,k_2} - e_{j,k_1} - 1 = (k_2 - k_1)(M(2q+1)+2) - \delta_{k_1-P/2} + \delta_{k_2-P/2} + (2q+1)(j-i)/2 - 1 > q.$$

5) For any $0 \le k < P/2$, we have $e_{i,k} - e_{j,k+P/2} = N - (M(2q+1)+2) + \delta_k - \delta_{k+1} + 1 + q + (2q+1)(i+j)/2 > q$, and $e_{i,k+P/2} - e_{j,k} - 1 = M(2q+1) + 1 - q + \delta_{k+1} - \delta_k - (2q+1)(i+j)/2 > q$.

Now we prove the case of i even and j odd.

We omit the proof for $P = 2$ since it is straightforward.

For $P > 2$, we consider the differences of $e_{i,k_1} - e_{j,k_2}$ and $e_{i,k_2} - e_{j,k_1} - 1$ for the following 6 cases of (k_1, k_2): 1) $0 \le k_1 < k_2 < P/2$; 2) $0 \le k_1 < P/2 < k_2 < P$ with $k_1 < k_2 - P/2$; 3)

$0 \le k_1 < P/2 < k_2 < P$ with $k_1 > k_2 - P/2$; 4) $P/2 < k_1 < k_2 < P$; 5) $k_2 = k_1 + P/2, i > j$; and 6) $k_2 = k_1 + P/2, i < j$.

1) For any $0 \le k_1 < k_2 < P/2$, we have

$$e_{i,k_1} - e_{j,k_2} = N - (M(2q+1)+2) + \delta_{k_1} - \delta_{k_2+1} + 1 + (2q+1)(i+j+1)/2.$$

By Lemma 4, it follows that

$$e_{i,k_1} - e_{j,k_2} \ge P/2 - 1 + \delta_{k_1} - \delta_{k_2+1} + 1 + (2q+1)(i+j+1)/2 > q.$$

Moreover, we have

$$e_{i,k_2} - e_{j,k_1} - 1 = (M(2q+1)+2) - \delta_{k_1} + \delta_{k_2+1} - 1 - (2q+1)(i+j+1)/2 - 1 > q.$$

2) For any $0 \le k_1 < P/2 < k_2 < P$ with $k_1 < k_2 - P/2$, we have

$$e_{i,k_1} - e_{j,k_2} = N + (k_1 - k_2 + P/2)(M(2q+1)+2) + \delta_{k_1} - \delta_{k_2-P/2} + (2q+1)(i-j+1)/2 - q - 1 > q.$$

By Lemma 4, it follows that

$$e_{i,k_1} - e_{j,k_2} \ge (M(2q+1)+2) + P/2 - 1 + \delta_{k_1} - \delta_{k_2-P/2} + (2q+1)(i-j+1)/2 - q - 1 > q.$$

Moreover, we have

$$e_{i,k_2} - e_{j,k_1} - 1 = (k_2 - P/2 - k_1)(M(2q+1)+2) - \delta_{k_1} + \delta_{k_2-P/2} - (2q+1)(i-j+1)/2 + q + 1 > q.$$

3) For any $0 \le k_1 < P/2 < k_2 < P$ with $k_1 > k_2 - P/2$, we have

$$e_{i,k_1} - e_{j,k_2} = (k_1 - k_2 + P/2)(M(2q+1)+2) + \delta_{k_1} - \delta_{k_2-P/2} + (2q+1)(i-j+1)/2 - q - 1 > q.$$

Moreover, we have

$$e_{i,k_2} - e_{j,k_1} - 1 = N + (k_2 - P/2 - k_1)(M(2q+1)+2) - \delta_{k_1} + \delta_{k_2-P/2} - (2q+1)(i-j+1)/2 + q + 1 > q.$$

By Lemma 4, it follows that

$$e_{i,k_2} - e_{j,k_1} - 1 \ge (M(2q+1)+2) + P/2 - 1 - \delta_{k_1} + \delta_{k_2-P/2} - (2q+1)(i-j+1)/2 + q + 1 > q.$$

4) For any $P/2 < k_1 < k_2 < P$, we have

$$e_{i,k_1} - e_{j,k_2} = N - (M(2q+1)+2) + \delta_{k_1-P/2} - \delta_{k_2-P/2+1} + 1 + (2q+1)(i+j+1)/2.$$

By Lemma 4, it follows that

$$e_{i,k_1} - e_{j,k_2} \ge P/2 - 1 + \delta_{k_1-P/2} - \delta_{k_2-P/2+1} + 1 + (2q+1)(i+j+1)/2 > q.$$

Moreover, we have

$$e_{i,k_2} - e_{j,k_1} - 1 = (M(2q+1)+2) - \delta_{k_1-P/2} + \delta_{k_2-P/2+1} - 1 - (2q+1)(i+j+1)/2 - 1 > q.$$

5) For any $0 \le k < P/2$ and $i > j$, we have $e_{i,k} - e_{j,k+P/2} = (2q+1)(i-j+1)/2 - q - 1 \ge q$, and $e_{i,k+P/2} - e_{j,k} - 1 = N - (2q+1)(i-j+1)/2 + q > q$.

6) For any $0 \le k < P/2$ and $i < j$, we have $e_{i,k} - e_{j,k+P/2} = N + (2q+1)(i-j+1) - q - 1 > q$, and $e_{i,k+P/2} - e_{j,k} - 1 = (2q+1)(j-i-1)/2 + q \ge q$.

Thus, by the analysis above, $\min(\mathbf{d}_{P/2}) = q$, and for any $1 \le i \le P-1$ with $i \ne P/2$, $\min(\mathbf{d}_i) > q$. □

Proof of Theorem 3: By Theorem 1, Lemmas 5, 6, 7, and 8, the result follows. □

5 Optimal Sequence Sets

In this section, we derive the conditions under which the new sequence sets are optimal.

Lemma 9 ([10]) *For any LCZ sequence set with parameters (v, t, L_{CZ}, δ), the following bound holds:*

$$t \cdot L_{CZ} - 1 \le \frac{v-1}{1-\delta^2/v}. \tag{13}$$

In particular, if $\delta = 0$, the bound (13) can be simplified as

$$t \le \left\lfloor \frac{v}{L_{CZ}} \right\rfloor. \tag{14}$$

5.1 The Case of $P|L$

Theorem 4 *Let $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_{M-1}\}$ be the set of shift sequences defined by (6) or (7). Let σ be defined by (5), and $N = ML + P + \sigma + r$ with $0 \le r < L$. If $\delta = 0$ and $L > P(P + \sigma + r)$, then $\mathcal{S}$ constructed in Section 3 is an optimal ZCZ sequence set with parameters (NP, MP, L).*

Proof. By Theorem 2, $\mathcal{S}$ is a ZCZ sequence set with parameters (NP, MP, L). Because

$$\left\lfloor \frac{NP}{L} \right\rfloor = \left\lfloor \frac{(ML+P+\sigma+r)P}{L} \right\rfloor = \left\lfloor MP + \frac{(P+\sigma+r)P}{L} \right\rfloor = MP,$$

$\mathcal{S}$ is an optimal ZCZ sequence set with parameters (NP, MP, L) by (14). □

Example 4 *Let $P = 3, L = 15, N = 79$. Then $M = 5$, and the shift sequence $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ is defined as follows: $\mathbf{e}_0 = \{0, 47, 53\}, \mathbf{e}_1 = \{5, 42, 58\}, \mathbf{e}_2 = \{10, 37, 63\}, \mathbf{e}_3 = \{15, 32, 68\}$, and $\mathbf{e}_4 =$*

$\{20, 27, 73\}$. *One can check that $\mathcal{E}$ satisfies Lemmas 1, 2, and 3. Let*

$$\begin{bmatrix} \omega_{0,0} & \omega_{0,1} & \omega_{0,2} \\ \omega_{1,0} & \omega_{1,1} & \omega_{1,2} \\ \omega_{2,0} & \omega_{2,1} & \omega_{2,2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix},$$

where $i^2 = -1$. Thus, based on any complex-valued perfect sequence with period 79, we can construct an optimal ZCZ sequence set with parameters $(237, 15, 15)$.

Theorem 5 *Let $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_{M-1}\}$ be the set of shift sequences defined by (6) or (7). Let σ be defined by (5), and $N = ML + P + \sigma + r$ with $0 \le r < L$. If $\delta = 1$ and $L > P(3P + r + \sigma)$, then $\mathcal{S}$ constructed in Section 3 is an optimal LCZ sequence set with parameters (NP, MP, L, P).*

Proof. By Theorem 2, $\mathcal{S}$ is an LCZ sequence set with parameters (NP, MP, L, P). Because $L > P(3P + r + \sigma)$, we have

$$\begin{aligned} (ML + L/P)(N - P) &= (N - P - \sigma - r + L/P)(N - P) \\ &\ge (N + 2P)(N - P) \\ &= N^2 + NP - 2P^2 \\ &> N^2. \end{aligned}$$

Hence

$$0 < N^2 - 1 - ML(N - P) < (N - P)L/P.$$

By (13),

$$MP \cdot L \le \frac{NP - 1}{1 - P^2/(NP)} + 1 = \frac{(N^2 - 1)P}{N - P}.$$

On the other hand,

$$\left\lfloor \frac{(N^2 - 1)P}{(N - P)L} \right\rfloor = \left\lfloor MP + \frac{(N^2 - 1 - ML(N - P))P}{(N - P)L} \right\rfloor = MP.$$

Hence $\mathcal{S}$ is an optimal LCZ sequence set with parameters (NP, MP, L, P). □

Example 5 *Let $P = 3, L = 33, N = 103$. Then $M = 3$, and the shift sequence $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\}$ is defined as follows: $\mathbf{e}_0 = \{0, 57, 69\}, \mathbf{e}_1 = \{11, 46, 80\}$, and $\mathbf{e}_2 = \{22, 35, 91\}$. One can check that $\mathcal{E}$ satisfies Lemmas 1, 2, and 3. Let*

$$\begin{bmatrix} \omega_{0,0} & \omega_{0,1} & \omega_{0,2} \\ \omega_{1,0} & \omega_{1,1} & \omega_{1,2} \\ \omega_{2,0} & \omega_{2,1} & \omega_{2,2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i/3} & e^{4\pi i/3} \\ 1 & e^{4\pi i/3} & e^{2\pi i/3} \end{bmatrix},$$

where $i^2 = -1$. *Thus, based on the (complex-valued) Legendre sequence with period* 103*, we can construct an optimal LCZ sequence set with parameters* $(309, 9, 33, 3)$.

5.2 The Case of P Even, and $L \equiv P/2 \pmod{P}$

Theorem 6 *Let* $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_{M-1}\}$ *be the set of shift sequences defined by (9) or (10). Let* σ *be defined by (8), and* $N = ML + 3P/2 - 1 + \sigma + r$ *with* $0 \leq r < L$. *If* $\delta = 0$ *and* $L > P(3P/2 - 1 + \sigma + r)$, *then* $\mathcal{S}$ *constructed in Section 3 is an optimal ZCZ sequence set with parameters* (NP, MP, L).

Proof. The proof is similar to that of Theorem 4, so we omit it. □

Example 6 *Let* $P = 4, L = 22, N = 115$. *Then* $M = 5$, *and the shift sequence* $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ *is defined as follows:* $\mathbf{e}_0 = \{0, 57, 51, 109\}, \mathbf{e}_1 = \{45, 103, 6, 63\}, \mathbf{e}_2 = \{11, 68, 40, 98\}, \mathbf{e}_3 = \{34, 92, 17, 74\}$, *and* $\mathbf{e}_4 = \{22, 79, 29, 87\}$. *One can check that* $\mathcal{E}$ *satisfies Lemmas 5, 6, 7, and 8. Let*

$$\begin{bmatrix} \omega_{0,0} & \omega_{0,1} & \omega_{0,2} & \omega_{0,3} \\ \omega_{1,0} & \omega_{1,1} & \omega_{1,2} & \omega_{1,3} \\ \omega_{2,0} & \omega_{2,1} & \omega_{2,2} & \omega_{2,3} \\ \omega_{3,0} & \omega_{3,1} & \omega_{3,2} & \omega_{3,3} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix},$$

where $i^2 = -1$. *Thus, based on any complex-valued perfect sequence with period* 115*, we can construct an optimal ZCZ sequence set with parameters* $(460, 20, 22)$.

Theorem 7 *Let* $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, ..., \mathbf{e}_{M-1}\}$ *be the set of shift sequences defined by (9) or (10). Let* σ *be defined by (8), and* $N = ML + 3P/2 - 1 + \sigma + r$ *with* $0 \leq r < L$. *If* $\delta = 1$ *and* $L > P(7P/2 - 1 + \sigma + r)$, *then* $\mathcal{S}$ *constructed in Section 3 is an optimal LCZ sequence set with parameters* (NP, MP, L, P).

Proof. The proof is similar to that of Theorem 5, so we omit it. □

Example 7 *Let* $P = 2, L = 31, N = 127$. *Then* $M = 4$, *and the shift sequence* $\mathcal{E} = \{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *is defined as follows:* $\mathbf{e}_0 = \{0, 110\}, \mathbf{e}_1 = \{94, 16\}, \mathbf{e}_2 = \{31, 79\}$, *and* $\mathbf{e}_3 = \{63, 47\}$. *One can check that* $\mathcal{E}$ *satisfies Lemmas 5, 6, 7, and 8. Let*

$$\begin{bmatrix} \omega_{0,0} & \omega_{0,1} \\ \omega_{1,0} & \omega_{1,1} \end{bmatrix} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix},$$

where $i^2 = -1$. *Thus, based on any complex-valued two-level sequence with period* 127*, we can construct an optimal LCZ sequence set with parameters* $(254, 8, 31, 2)$.

6 Conclusion

In this paper, we present a new general construction of LCZ or ZCZ sequence sets based on interleaving techniques. The construction in [12] is a special case of this new construction (It should be pointed out that in this case the shift sequence $\{\mathbf{e}_0, \mathbf{e}_1\}$ is different from that in [12].). In two cases, we present the shift sequences. The conditions are also derived under which the new sequence sets are optimal. In order to specify this new construction, we present some examples. More shift sequences are desirable.

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