Even Periodic and Odd Periodic Complementary Sequence Pairs from Generalized Boolean Functions

Yang Yang[1,2,3] Xiaohu Tang[1] and Guang Gong[2]

[1]Institute of Mobile Communications, Southwest Jiaotong University
Chengdu, 610031, P.R. CHINA

[2]Department of Electrical and Computer Engineering University of Waterloo
Waterloo, Ontario N2L 3G1, CANADA

Email: yang_data@yahoo.cn, xhutang@ieee.org,
ggong@calliope.uwaterloo.ca

Abstract

A pair of two sequences is called the even periodic (odd periodic) complementary sequence pair if the sum of their even periodic (odd periodic) correlation function is a delta function. The well-known Golay aperiodic complementary sequence pair (Golay pair) is a special case of even periodic (odd periodic) complementary sequence pair. In this paper, we presented several classes of even periodic and odd periodic complementary pairs based on the generalized Boolean functions, but which do not form Gloay pairs. The proposed sequences could be used to design signal sets, which has been applied in direct sequence code division multiple (DS-CDMA) cellular communication systems.

Index Terms. Even periodic complementary sequence pair, odd periodic complementary sequence pair, Golay complementary pair, Golay sequences, generalized Boolean function.

1 Introduction

Correlation is a measure of the similarity or relatedness between two sequences. Sequences with low autocorrelation properties have been widely used in modern communications, radar, sonar, and in the field of measuring techniques [1, 11, 21]. Generally speaking, the sequences with low aperiodic, perfect periodic, or odd perfect periodic autocorrelation property are very favorable [8], [16]-[19]. However, it seems extremely hard to design such sequences. Therefore people reduced the requirement in two

[3]Yang Yang is a visiting Ph. D. student (Oct. 2010- Sep. 2012) in the Department of Electrical and Computer Engineering, University of Waterloo.

directions: (1) design single sequence with autocorrelation property as low as possible; (2) construct a pair or a set of sequences such that the sum of their autocorrelation functions is a delta function.

In [5, 6], a pair of binary sequences was first introduced by Golay in connection with infrared multislit spectrometry. Such sequences pairs have a property that the sum of their respective aperiodic autocorrelation function is a delta function. For short, in the literature the pair is called Golay pair and each sequence is called a Golay sequence. Later, the aperiodic complementary pair was generalized to aperiodic sequence set, and also periodic or odd periodic complementary sequence pair [23], [12], [2], [9], [24]. All these sequences have found important applications in the practical systems. For example, Golay sequences are used in fields such as in multislit spectrometry, ultrasound measurements, acoustic measurements, radar pulse compression, Wi-Fi networks, 3G CDMA wireless networks, and OFDM communication systems [9]. Periodic complementary sets are used to design a sets of signature vectors, which have been used in direct sequence code division multiple (DS-CDMA) cellular communication systems [10, 4].

Note that the Golay pairs are a special class of even periodic or odd periodic complementary pairs. So there would be interesting to find out two sequences, which is even periodic or odd periodic complementary pair although they do not form a Golay pair. In this paper, our motivation is to design such odd periodic and periodic complementary pair by means of the generalized Boolean function technique, based on which Davis and Jedwab have developed an elegant description for a large class of Golay pairs in [2].

This paper is organized as follows. In Section 2, we will provide the necessary background information and notions used throughout the paper. In Sections 3 and 4, we will show the constructions of even periodic and odd periodic complementary sequence pairs by using generalized Boolean functions.

2 Preliminaries

Let N be a positive integer, and $a = (a_0, a_1, \cdots, a_{N-1})$ be a complex sequence of period N. The *aperiodic autocorrelation* function of a at shift τ, $0 \leq \tau \leq N-1$, is defined as

$$C_a(\tau) = \sum_{i=0}^{N-1-\tau} a_i a_{i+\tau}^* \tag{1}$$

where x^* denotes the conjugate of the complex number x. Based on it, the *periodic autocorrelation function* and *odd periodic autocorrelation function* of a at shift τ, $0 < \tau < N$, are respectively defined by

$$R_a(\tau) = C_a(\tau) + C_a(N-\tau)^* \tag{2}$$
$$\hat{R}_a(\tau) = C_a(\tau) - C_a(N-\tau)^* \tag{3}$$

Let a and b be two sequences of length N. A pair of a and b is said to be an *aperiodic complementary sequences pair* if $C_a(\tau) + C_b(\tau) = 0$ for $0 < \tau < N$. In this case, a and b are also called the Golay sequences, and the pair is called a Golay pair. Similarly, (a, b) is called *periodic complementary pair* (or *odd periodic complementary pair*) if $R_a(\tau) + R_b(\tau) = 0$ (or $\hat{R}_a(\tau) + \hat{R}_b(\tau) = 0$) for any $0 < \tau < N$.

Let H be a positive integer and ξ be the primitive element of H-th unity, i.e., $\xi = \exp(2\pi\sqrt{-1}/H)$. A modulated sequence of sequence $(a_0, a_1, \cdots, a_{N-1})$ over Z_H is written as the complex sequence $(\xi^{a_0}, \xi^{a_1}, \cdots, \xi^{a_{N-1}})$. In the following, we will use those two expressions of sequences interchangeably for convenience.

A generalized Boolean function $f(x_1, \cdots, x_m)$ from $\mathbb{Z}_2^m$ to $\mathbb{Z}_H$ is represented as a linear combination of the 2^m monomials:

$$f(x_1, \cdots, x_m) = \sum_{I \in \{1, \cdots, m\}} a_I \prod_{i \in I} x_i, \ a_I \in Z_H.$$

Let $(i_1, i_2, \cdots, i_m)$ be the binary representation of the integer $i = \sum_{k=1}^{m} 2^{m-k} i_k$. Then a sequence over Z_H of period 2^m can be generated from the truth table of a Boolean function $f(x_1, x_2, \cdots, x_m)$ from $\mathbb{Z}_2^m$ to $\mathbb{Z}_H$: the i-th element of the sequence is $f(i_1, i_2, \cdots, i_m)$. For example, $m = 3$, $H = 2$, and $f(x_1, x_2, x_3) = x_1 x_2 + x_3$, the sequence is defined as

$$(f(0,0,0), f(0,0,1), f(0,1,0), f(0,1,1), f(1,0,0), f(1,0,1), f(1,1,0), f(1,1,1)),$$

i.e., $(0,1,0,1,0,1,1,0)$, which can be regraded as $(1,-1,1,-1,1,-1,-1,1)$.

In [2], Davis and Jedwab developed a powerful theory to construct Golay sequence in terms of generalized Boolean function. From now on, it is always assumed that $m \geq 4$ is an integer and π is a permutation from $\{1, \cdots, m\}$ to itself.

Definition 1 *Define a sequence* $a = \{a_i\}_{i=0}^{2^m-1}$ *over* $\mathbb{Z}_H$*, whose elements are given by*

$$a_i = \frac{H}{2} \sum_{k=1}^{m-1} i_{\pi(k)} i_{\pi(k+1)} + \sum_{k=1}^{m} c_k i_k + c_0, \tag{4}$$

where $c_i \in \mathbb{Z}_H$*,* $i = 0, 1, \cdots, m$.

When $H = 2^h$, $h \geq 1$ an integer, Davis and Jedwab proved that $\{a_i\}$ and $\{a_i + 2^{h-1} i_{\pi(1)} + c'\}$ form a Golay pair for any $c' \in \mathbb{Z}_{2^h}$ in the Theorem 3 of [2]. Later on, Paterson generalized this result by replacing $\mathbb{Z}_{2^h}$ with $\mathbb{Z}_H$ [15], where $H \geq 2$ is an arbitrary even integer.

Fact 1 (Corollary 11, [15]) *Let* $a = \{a_i\}_{i=0}^{2^m-1}$ *be the sequence given in Definition 1. Then the pair of the sequences* a_i *and* $a_i + \frac{H}{2} i_{\pi(1)} + c'$ *form a Golay complementary pair for any* $c' \in \mathbb{Z}_H$.

In the next two sections, we will present periodic and odd periodic complementary sequence pairs by using other generalized Boolean functions.

3 Even Periodic Complementary Pairs

In this section, we will consider the pair of sequences $a = \{a_i\}_{i=0}^{2^m-1}$ and $b = \{b_i\}_{i=0}^{2^m-1}$, where

$$\begin{cases} a_i &= \frac{H}{2}\sum_{k=1}^{m-1} i_{\pi(k)}i_{\pi(k+1)} + \sum_{k=1}^{m} c_k i_{\pi(k)} + c_0 + \frac{H}{2} i_{\pi(p)}i_{\pi(q)} \\ b_i &= a_i + \frac{H}{2}\ell_i \end{cases} \quad (5)$$

where p and q are two integers with $1 \le p < q \le m$, and ℓ_i is a linear function from $\mathbb{Z}_2^m$ to $\mathbb{Z}_H$, H an even integer.

3.1 Results

Theorem 1 *Let $q = p+2$ with $1 \le p \le m-2$. The pair of sequences a and b given by (5) is a periodic complementary pair, if*

1. $p = 1$,
   - (a) $\pi(2) = 1$, $c_1 \in \{0, H/2\}$, *and* $\ell_i \in \{i_{\pi(1)}, i_{\pi(3)}, i_{\pi(m)}, i_{\pi(1)} + i_{\pi(3)} + i_{\pi(m)}\}$;
   - (b) $\pi(1) = 1$, $c_1 \in \{0, H/2\}$, *and* $\ell_i \in \{i_{\pi(2)}, i_{\pi(3)}, i_{\pi(m)}, i_{\pi(2)} + i_{\pi(3)} + i_{\pi(m)}\}$;
2. $2 \le p \le m-2$, $\pi(p+1) = 1$, $c_1 \in \{0, H/2\}$, *and* $\ell_i \in \{i_{\pi(1)}, i_{\pi(m)}, i_{\pi(1)} + i_{\pi(p)} + i_{\pi(p+2)}, i_{\pi(p)} + i_{\pi(p+2)} + i_{\pi(m)}\}$.

Theorem 2 *Let $p = 2$ and $q = 3$. The pair of sequences a and b given by (5) is a periodic complementary pair if $\ell_i \in \{d_0 i_{\pi(1)} + d_1 i_{\pi(2)} + i_{\pi(3)}, d_0 i_{\pi(1)} + d_1 i_{\pi(2)} + i_{\pi(m)} : d_0, d_1 = 0, 1\}$, $c_1 \in \{0, H/2\}$, $\pi(2) = 1$ or $\pi(1) = 1$.*

Theorem 3 *Let $p = 2$, and $q = m$. The pair of sequences a and b given by (5) is a periodic complementary pair if $\ell_i \in \{d_0 i_{\pi(2)} + i_{\pi(m)}, d_0 i_{\pi(2)} + i_{\pi(3)}, i_{\pi(1)} + d_0 i_{\pi(2)} + i_{\pi(w)}, i_{\pi(1)} + d_0 i_{\pi(2)} + i_{\pi(3)} + i_{\pi(w)} + i_{\pi(m)} : d_0 = 0, 1\}$, $c_1 \in \{0, H/2\}$, and $\pi(1) = 1$.*

Define a mapping $\pi'(k) = \pi(m+1-k)$, $k \in \{1, \cdots, m\}$, and replace π by π', the following corollary follows from Theorems 1-3.

Corollary 1 *The pair of sequences a and b given by (5) is a periodic complementary pair, if*

1. $p = m-2$, $q = m$,
   - (a) $\pi(m-1) = 1$, $c_1 \in \{0, H/2\}$, *and* $\ell_i \in \{i_{\pi(m)}, i_{\pi(m-2)}, i_{\pi(1)}, i_{\pi(m)} + i_{\pi(m-2)} + i_{\pi(1)}\}$;
   - (b) $\pi(m) = 1$, $c_1 \in \{0, H/2\}$, *and* $\ell_i \in \{i_{\pi(m-1)}, i_{\pi(m-2)}, i_{\pi(1)}, i_{\pi(m-1)} + i_{\pi(m-2)} + i_{\pi(1)}\}$;
2. $3 \le p \le m-2$, $q = p+2$, $\pi(m-p) = 1$, $c_1 \in \{0, H/2\}$, *and* $\ell_i \in \{i_{\pi(1)}, i_{\pi(m)}, i_{\pi(m)} + i_{\pi(m+1-p)} + i_{\pi(m-1-p)}, i_{\pi(m+1-p)} + i_{\pi(m-1-p)} + i_{\pi(1)}\}$.

3. $p=m-2$ and $q=m-1$, $\ell_i \in \{d_0 i_{\pi(m)}+d_1 i_{\pi(m-1)}+i_{\pi(m-2)}, d_0 i_{\pi(m)}+d_1 i_{\pi(m-1)}+i_{\pi(1)} : d_0, d_1 = 0,1\}$, $c_1 \in \{0, H/2\}$, $\pi(m-1)=1$ or $\pi(m)=1$.

4. $p=1$, and $q=m-1$, $\ell_i \in \{d_0 i_{\pi(m-1)}+i_{\pi(1)}, d_0 i_{\pi(m-1)}+i_{\pi(m-2)}, i_{\pi(m)}+d_0 i_{\pi(m-1)}+i_{\pi(1)}, i_{\pi(m)}+d_0 i_{\pi(m-1)}+i_{\pi(m-2)} : d_0 = 0,1\}$, $c_1 \in \{0, H/2\}$, and $\pi(m)=1$.

Remark 1 *The two sequences a and b given by (5) are different from the Golay sequences given by Fact 1.*

Example 1 *Let $m=5$, $H=2$ and $\pi=(23154)$ be the permutation. Let $a_i = \sum_{k=1}^{4} i_{\pi(k)} i_{\pi(k+1)} + i_{\pi(2)} i_{\pi(4)}$ and $b_i = a_i + i_{\pi(1)}$. The periodic autocorrelation function of binary sequences a and b are listed in Table 1. Furthermore they form a periodic complementary sequence pair.*

Table 1: The autocorrelation function of binary sequences a and b

a	$(0,0,0,1,0,1,0,0,0,0,0,1,1,0,1,1,0,1,0,0,1,1,1,0,0,1,0,0,0,0,0,1)$
b	$(0,0,0,1,0,1,0,0,1,1,1,0,0,1,0,0,0,1,0,0,1,1,1,0,1,0,1,1,1,1,1,0)$
$\{R_a(\tau)\}_{\tau=1}^{31}$	$(0,0,4,0,0,8,-4,0,4,8,0,0,-4,0,0,0,0,0,-4,0,$ $0,8,4,0,-4,8,0,0,4,0,0)$
$\{R_b(\tau)\}_{\tau=1}^{31}$	$(0,0,-4,0,0,-8,4,0,-4,-8,0,0,4,0,0,0,0,0,4,0,0,$ $-8,-4,0,4,-8,0,0,-4,0,0)$

Example 2 *Let $m=5$, $H=4$ and $\pi=(23154)$ be the permutation. Let $a_i = 2\sum_{k=1}^{4} i_{\pi(k)} i_{\pi(k+1)} + 2i_{\pi(2)} i_{\pi(4)} + i_{\pi(5)}$ and $b_i = a_i + 2i_{\pi(1)}$. The periodic autocorrelation function of quaternary sequences a and b are listed in Table 2. Furthermore they form an odd periodic complementary sequence pair. In Table 2 and Table 4 below, $\xi = \sqrt{-1}$.*

Table 2: The autocorrelation function of quaternary sequences a and b

a	$(0,0,1,3,0,2,1,1,0,0,1,3,2,0,3,3,0,2,1,1,2,2,3,1,0,2,1,1,0,0,1,3)$
b	$(0,0,1,3,0,2,1,1,2,2,3,1,0,2,1,1,0,2,1,1,2,2,3,1,2,0,3,3,2,2,3,1)$
$\{R_a(\tau)\}_{\tau=1}^{31}$	$(0,0,4\xi,0,0,8\xi,4\xi,0,4\xi,-8\xi,0,0,4\xi,0,0,0,0,0,-4\xi,$ $0,0,8\xi,-4\xi,0,-4\xi,-8\xi,0,0,-4\xi,0,0)$
$\{R_b(\tau)\}_{\tau=1}^{31}$	$(0,0,-4\xi,0,0,-8\xi,-4\xi,0,-4\xi,8\xi,0,0,-4\xi,0,0,0,0,0,4\xi,$ $0,0,-8\xi,4\xi,0,4\xi,8\xi,0,0,4\xi,0,0)$

3.2 Proof of Results

From now on by convenience, for any given integers $1 \le \tau < 2^m - 1$, $0 \le i < 2^m - \tau$ and $0 \le s < \tau$, we always set $j = i + \tau$, $t = s + 2^m - \tau$, and let $(i_1, i_2, \cdots, i_m)$, $(j_1, j_2, \cdots, j_m)$, $(s_1, s_2, \cdots, s_m)$ and $(t_1, t_2, \cdots, t_m)$ be the binary representations of i, j, s and t, respectively.

In this section, we will prove that

$$R_a(\tau)+R_b(\tau)=0,\quad 0<\tau<2^m$$

for pair (a,b) given in (5).

Due to relationship between aperiodic autocorrelation and even periodic (resp. odd periodic) autocorrelation shown in (2) (resp. (3)), we first compute

$$\begin{aligned} C_a(\tau)+C_b(\tau) &= \sum_{i=0}^{2^m-1-\tau}(\xi^{a_i-a_j}+\xi^{b_i-b_j}) \\ &= \sum_{i=0}^{2^m-1-\tau}\xi^{a_i-a_j}(1+\xi^{\frac{H}{2}(\ell_i-\ell_j)}) \\ &= \sum_{i=0}^{2^m-1-\tau}\xi^{a_i-a_j}(1+(-1)^{\ell_i-\ell_j}) \\ &= 2\sum_{i\in\mathbf{J}(\tau)}\xi^{a_i-a_j} \end{aligned}$$

where $j=i+\tau$ and the set $\mathbf{J}(\tau)$ is defined as

$$\mathbf{J}(\tau)=\{0\le i<2^m-\tau : \ell_i=\ell_j\}.$$

Proof of Theorem 1: We only prove for $p=1$ since $2\le p\le m-2$ can be proved in the same fashion. We divide $\mathbf{J}(\tau)$ into the following four disjoint subsets

$$\begin{aligned} \mathbf{J}_1(\tau) &= \{0\le i<2^m-\tau : i_{\pi(1)}+i_{\pi(3)}\ne j_{\pi(1)}+j_{\pi(3)}, i_{\pi(2)}=j_{\pi(2)}, \ell_i=\ell_j\}, \\ \mathbf{J}_2(\tau) &= \{0\le i<2^m-\tau : i_{\pi(1)}+i_{\pi(3)}\ne j_{\pi(1)}+j_{\pi(3)}, i_{\pi(2)}\ne j_{\pi(2)}, \ell_i=\ell_j\}, \\ \mathbf{J}_3(\tau) &= \{0\le i<2^m-\tau : i_{\pi(1)}+i_{\pi(3)}= j_{\pi(1)}+j_{\pi(3)}, i_{\pi(1)}=j_{\pi(1)}, \ell_i=\ell_j\}, \\ \mathbf{J}_4(\tau) &= \{0\le i<2^m-\tau : i_{\pi(1)}+i_{\pi(3)}= j_{\pi(1)}+j_{\pi(3)}, i_{\pi(1)}\ne j_{\pi(1)}, \ell_i=\ell_j\}. \end{aligned}$$

For any $i\in\mathbf{J}_1(\tau)$, let i' and j' be the two integers whose bits in the binary representation are defined by

$$i'_{\pi(k)}=\begin{cases} i_{\pi(k)}, & \text{if } k\ne 2 \\ 1-i_{\pi(k)}, & \text{if } k=2 \end{cases} \tag{6}$$

and

$$j'_{\pi(k)}=\begin{cases} j_{\pi(k)}, & \text{if } k\ne 2 \\ 1-j_{\pi(k)}, & \text{if } k=2. \end{cases} \tag{7}$$

In other words, i' and j' are obtained from i and j by "flipping" the 2-th bit in $(i_{\pi(1)},\cdots,i_{\pi(m)})$ and $(j_{\pi(1)},\cdots,j_{\pi(m)})$. Then, the following facts hold.

1. $i\to i'$ and $j\to j'$ are respectively one-to-one mapping;

2. $j' - i' = j - i = \tau$;

3. $\ell_{i'} = \ell_{j'}$ for $i \in \mathbf{J}_1(\tau)$;

4. $i' \in \mathbf{J}_1(\tau)$.

Given $i \in \mathbf{J}_1(\tau)$, by (5) we have

$$(a_i - a_j) - (a_{i'} - a_{j'}) = \frac{H}{2}(i_{\pi(1)} + j_{\pi(1)} + i_{\pi(3)} + j_{\pi(3)}) = \frac{H}{2},$$

which indicates that $\xi^{a_i - a_j}/\xi^{a_{i'} - a_{j'}} = -1$, i.e., $\xi^{a_i - a_j} + \xi^{a_{i'} - a_{j'}} = 0$. Hence,

$$\begin{aligned}\sum_{i \in \mathbf{J}_1(\tau)} \xi^{a_i - a_j} &= \frac{H}{2} \sum_{i \in \mathbf{J}_1(\tau)} \xi^{a_i - a_j} + \frac{H}{2} \sum_{i' \in \mathbf{J}_1(\tau)} \xi^{a_{i'} - a_{j'}} \\ &= \frac{H}{2} \sum_{i \in \mathbf{J}_1(\tau)} \left(\xi^{a_i - a_j} + \xi^{a_{i'} - a_{j'}}\right) = 0\end{aligned}$$

and then

$$\sum_{i \in \mathbf{J}_1(2^m - \tau)} \xi^{a_i - a_j} = 0$$

by replacing $2^m - \tau$ with τ.

For any $i \in \mathbf{J}_3(\tau)$, since $j \neq i$, we can define v as follows:

$$v = \min\{1 \leq k \leq m : i_{\pi(k)} \neq j_{\pi(k)}\}. \tag{8}$$

Obviously, $2 \leq v \leq m$. In addition, $v \neq 3$ because $i_{\pi(3)} = j_{\pi(3)}$ from the definition of $\mathbf{J}_3(\tau)$. For any $i \in \mathbf{J}_3(\tau)$, let i' and j' be the two integers whose bits in the binary representation are defined by

$$i'_{\pi(k)} = \begin{cases} i_{\pi(k)}, & \text{if } k \neq v - 1 \\ 1 - i_{\pi(k)}, & \text{if } k = v - 1 \end{cases} \tag{9}$$

and

$$j'_{\pi(k)} = \begin{cases} j_{\pi(k)}, & \text{if } k \neq v - 1 \\ 1 - j_{\pi(k)}, & \text{if } k = v - 1. \end{cases} \tag{10}$$

Note that the following facts hold.

1. $i \to i'$ and $j \to j'$ are respectively one-to-one mapping;

2. $j' - i' = j - i = \tau$;

3. $\ell_{i'} = \ell_{j'}$ for $i \in \mathbf{J}_3(\tau)$;

4. $i' \in \mathbf{J}_3(\tau)$.

Consequently, we get from (5) that

$$
\begin{aligned}
&(a_i - a_j) - (a_{i'} - a_{j'}) \\
= &\begin{cases} \frac{H}{2}(i_{\pi(2)} + j_{\pi(2)} + i_{\pi(3)} + j_{\pi(3)}) + 2c_{\pi(1)}(i_{\pi(1)} - j_{\pi(1)}), & \text{if } v = 2 \\ \frac{H}{2}(i_{\pi(1)} + j_{\pi(1)} + i_{\pi(2)} + j_{\pi(2)} + i_{\pi(4)} + j_{\pi(4)}) + 2c_{\pi(3)}(i_{\pi(3)} - j_{\pi(3)}), & \text{if } v = 4 \\ \frac{H}{2}(i_{\pi(v-2)} + j_{\pi(v-2)} + i_{\pi(v)} + j_{\pi(v)} + 2c_{\pi(v-1)}(i_{\pi(v-1)} - j_{\pi(v-1)}), & \text{if } v > 4 \end{cases}
\end{aligned}
$$

which is $\frac{H}{2}$ according to the definition of v. Then, it gives $\xi^{a_i - a_j} + \xi^{a_{i'} - a_{j'}} = 0$. So,

$$
\sum_{i \in \mathbf{J}_3(\tau)} \xi^{a_i - a_j} = \sum_{i \in \mathbf{J}_3(2^m - \tau)} \xi^{a_i - a_j} = 0
$$

In the following, firstly we define two mappings as follows.

Mapping 1: If ($\pi(2) = 1$, $i_{\pi(2)} \neq j_{\pi(2)}$, and $i_{\pi(1)} + i_{\pi(3)} \neq j_{\pi(1)} + j_{\pi(3)}$) or ($\pi(1) = 1$, $i_{\pi(1)} \neq j_{\pi(1)}$, and $i_{\pi(2)} + i_{\pi(3)} \neq j_{\pi(2)} + j_{\pi(3)}$), then set two integers s and t as

$$
s_{\pi(k)} = \begin{cases} i_{\pi(k)}, & \text{if } \pi(k) = 1 \\ j_{\pi(k)}, & \text{if } \pi(k) \neq 1 \end{cases} \tag{11}
$$

and

$$
t_{\pi(k)} = \begin{cases} j_{\pi(k)}, & \text{if } \pi(k) = 1 \\ i_{\pi(k)}, & \text{if } \pi(k) \neq 1. \end{cases} \tag{12}
$$

By (5), we have

$$
\begin{aligned}
&(a_i - a_j) - (a_t - a_s) \\
= &\begin{cases} \frac{H}{2}(i_{\pi(1)} + j_{\pi(1)} + i_{\pi(3)} + j_{\pi(3)})(i_{\pi(2)} + j_{\pi(2)}) + 2c_{\pi(2)}(i_{\pi(2)} - j_{\pi(2)}), & \text{if } \pi(2) = 1 \\ \frac{H}{2}(i_{\pi(2)} + j_{\pi(2)} + i_{\pi(3)} + j_{\pi(3)})(i_{\pi(1)} + j_{\pi(1)}) + 2c_{\pi(1)}(i_{\pi(1)} - j_{\pi(1)}), & \text{if } \pi(1) = 1 \end{cases} \\
= &\frac{H}{2}
\end{aligned}
$$

because of $c_{\pi(2)} \in \{0, H/2\}$ for $\pi(2) = 1$ and $c_{\pi(1)} \in \{0, H/2\}$ for $\pi(1) = 1$. That is,

$$
\xi^{a_i - a_j} + \xi^{a_t - a_s} = 0.
$$

Mapping 2: If $i_{\pi(3)} + i_{\pi(m)} \neq j_{\pi(3)} + j_{\pi(m)}$, then set i' and j' be the two integers defined by

$$
i'_{\pi(k)} = 1 - j_{\pi(k)}, \quad k = 1, \cdots, m \tag{13}
$$

and

$$
j'_{\pi(k)} = 1 - i_{\pi(k)}, \quad k = 1, \cdots, m. \tag{14}
$$

Therefore, we have that

$$
(a_i - a_j) - (a_{i'} - a_{j'}) = \frac{H}{2}(i_{\pi(3)} + j_{\pi(3)} + i_{\pi(m)} + j_{\pi(m)}) = \frac{H}{2},
$$

which leads to

$$\xi^{a_i-a_j}+\xi^{a_{i'}-a_{j'}}=0.$$

Next based on the two mappings above, we investigate $\mathbf{J}_2(\tau)$ and $\mathbf{J}_4(\tau)$ case by case according to $\pi(2)=1$ or $\pi(1)=1$, and distinct ℓ_i.

Case 1. $\pi(2)=1$.

For $i\in\mathbf{J}_2(\tau)$, define two integers s and t by (11) and (12) respectively in Mapping 1. It is easy to check that the following facts hold.

1. $j\to s$ and $i\to t$ are respectively one-to-one mapping;
2. $t-s=2^{m-1}(j_{\pi(2)}-i_{\pi(2)})+\sum_{k=1,k\neq 2}^{m}2^{m-\pi(k)}(i_{\pi(k)}-j_{\pi(k)})=2^m-(j-i)=2^m-\tau$;
3. $\ell_s=\ell_t$ for $i\in\mathbf{J}_2(\tau)$;
4. $s\in\mathbf{J}_2(2^m-\tau)$.

Then,

$$\sum_{i\in\mathbf{J}_2(\tau)}\xi^{a_i-a_j}+\sum_{s\in\mathbf{J}_2(2^m-\tau)}\xi^{a_t-a_s}=0$$

For $i\in\mathbf{J}_4(\tau)$, we have $i_{\pi(1)}\neq j_{\pi(1)}$ and $i_{\pi(3)}\neq j_{\pi(3)}$. If $\ell_i\in\{i_{\pi(1)},i_{\pi(3)}\}$, then set $\mathbf{J}_4(\tau)$ is an empty set. If $\ell_i\in\{i_{\pi(m)},i_{\pi(1)}+i_{\pi(3)}+i_{\pi(m)}\}$, then $i_{\pi(m)}=j_{\pi(m)}$, i.e., $i_{\pi(3)}+i_{\pi(m)}\neq j_{\pi(3)}+j_{\pi(m)}$. Set two integers i' and j' defined by (13) and (14) respectively in Mapping 2. It is easily checked the following facts.

1. $i\to i'$ and $j\to j'$ are respectively one-to-one mapping;
2. $j'-i'=j-i=\tau$;
3. $\ell_{i'}=\ell_{j'}$ for $i\in\mathbf{J}_4(\tau)$;
4. $i'\in\mathbf{J}_4(\tau)$.

Therefore,

$$\sum_{i\in\mathbf{J}_4(\tau)}\xi^{a_i-a_j}=\sum_{i\in\mathbf{J}_4(2^m-\tau)}\xi^{a_i-a_j}=0$$

Case 2. $\pi(1)=1$.

The set $\mathbf{J}_2(\tau)$ can be divided into two disjoint subsets:

$$\begin{aligned}\mathbf{J}_5(\tau)&=\{0\leq i<2^m-\tau: i_{\pi(1)}=j_{\pi(1)}, i_{\pi(2)}\neq j_{\pi(2)}, i_{\pi(3)}\neq j_{\pi(3)}, \ell_i=\ell_j\}\\ \mathbf{J}_6(\tau)&=\{0\leq i<2^m-\tau: i_{\pi(1)}\neq j_{\pi(1)}, i_{\pi(2)}\neq j_{\pi(2)}, i_{\pi(3)}=j_{\pi(3)}, \ell_i=\ell_j\}.\end{aligned}$$

The set $\mathbf{J}_4(\tau)$ can be divided into two disjoint subsets:

$$\begin{aligned}\mathbf{J}_7(\tau) &= \{0 \le i < 2^m - \tau : i_{\pi(1)} \neq j_{\pi(1)}, i_{\pi(2)} = j_{\pi(2)}, i_{\pi(3)} \neq j_{\pi(3)}, \ell_i = \ell_j\} \\ \mathbf{J}_8(\tau) &= \{0 \le i < 2^m - \tau : i_{\pi(1)} = j_{\pi(1)}, i_{\pi(2)} \neq j_{\pi(2)}, i_{\pi(3)} \neq j_{\pi(3)}, \ell_i = \ell_j\}.\end{aligned}$$

For $\mathbf{J}_6(\tau)$ and $\mathbf{J}_7(\tau)$, by means of Mapping 1 we can derive that

$$\sum_{i\in\mathbf{J}_6(\tau)} \xi^{a_i-a_j} + \sum_{s\in\mathbf{J}_6(2^m-\tau)} \xi^{a_t-a_s} = 0$$

and

$$\sum_{i\in\mathbf{J}_7(\tau)} \xi^{a_i-a_j} + \sum_{s\in\mathbf{J}_7(2^m-\tau)} \xi^{a_t-a_s} = 0$$

Both $\mathbf{J}_5(\tau)$ and $\mathbf{J}_8(\tau)$ are empty unless $\ell_i \in \{i_{\pi(m)}, i_{\pi(2)}+i_{\pi(3)}+i_{\pi(m)}\}$, which implies $i_{\pi(m)} = j_{\pi(m)}$. Then, by using Mapping 2, we can deduce that

$$\sum_{i\in\mathbf{J}_5(\tau)} \xi^{a_i-a_j} = \sum_{s\in\mathbf{J}_8(\tau)} \xi^{a_i-a_j} = 0$$

and then

$$\sum_{i\in\mathbf{J}_5(2^m-\tau)} \xi^{a_i-a_j} = \sum_{s\in\mathbf{J}_8(2^m-\tau)} \xi^{a_i-a_j} = 0$$

Therefore, we obtain

$$\begin{aligned}&\sum_{i\in\mathbf{J}_2(\tau)} \xi^{a_i-a_j} + \sum_{s\in\mathbf{J}_2(2^m-\tau)} \xi^{a_t-a_s} \\ =& \left[\sum_{i\in\mathbf{J}_5(\tau)} \xi^{a_i-a_j} + \sum_{s\in\mathbf{J}_5(2^m-\tau)} \xi^{a_t-a_s}\right] + \left[\sum_{i\in\mathbf{J}_6(\tau)} \xi^{a_i-a_j} + \sum_{s\in\mathbf{J}_6(2^m-\tau)} \xi^{a_t-a_s}\right] = 0\end{aligned}$$

and

$$\begin{aligned}&\sum_{i\in\mathbf{J}_4(\tau)} \xi^{a_i-a_j} + \sum_{s\in\mathbf{J}_4(2^m-\tau)} \xi^{a_t-a_s} \\ =& \left[\sum_{i\in\mathbf{J}_7(\tau)} \xi^{a_i-a_j} + \sum_{s\in\mathbf{J}_7(2^m-\tau)} \xi^{a_t-a_s}\right] + \left[\sum_{i\in\mathbf{J}_8(\tau)} \xi^{a_i-a_j} + \sum_{s\in\mathbf{J}_8(2^m-\tau)} \xi^{a_t-a_s}\right] = 0.\end{aligned}$$

Finally, according to the above discussion we then obtain

$$R_a(\tau) + R_b(\tau) = \sum_{k=1}^{4} \left[\sum_{i\in\mathbf{J}_k(\tau)} \xi^{a_i-a_j} + \sum_{s\in\mathbf{J}_k(2^m-\tau)} \xi^{a_t-a_s}\right] = 0.$$

□

Proof of Theorem 2: Similar to Theorem 1, we can prove Theorem 2 by dividing $\mathbf{J}(\tau)$ and then using the following mapping.

1. For $\mathbf{J}_1(\tau) = \{0 \leq i < 2^m - \tau : i_{\pi(1)} = j_{\pi(1)}, (i_{\pi(2)} \neq j_{\pi(2)} \text{ or } i_{\pi(3)} = j_{\pi(3)}), \ell_i = \ell_j\}$, define i' and j' by (9) and (10) respectively where the integer v is defined as (8);

2. For $\mathbf{J}_2(\tau) = \{0 \leq i < 2^m - \tau : i_{\pi(1)} = j_{\pi(1)}, i_{\pi(2)} = j_{\pi(2)}, i_{\pi(3)} \neq j_{\pi(3)}, \ell_i = \ell_j\}$, define i' and j' by (13) and (14) respectively. Note that: $\mathbf{J}_2(\tau)$ is an empty set if $\ell_i \in \{d_0 i_{\pi(1)} + d_1 i_{\pi(2)} + i_{\pi(3)} : d_i = 0, 1\}$; When $\ell_i \in \{d_0 i_{\pi(1)} + d_1 i_{\pi(2)} + i_{\pi(m)} : d_i = 0, 1\}$, we have $i_{\pi(m)} = j_{\pi(m)}$;

3. For $\mathbf{J}_3(\tau) = \{0 \leq i < 2^m - \tau : i_{\pi(1)} \neq i_{\pi(1)}, j_{\pi(2)} \neq j_{\pi(2)}\}$, define s and t by (11) and (12) respectively;

4. For $\mathbf{J}_4(\tau) = \{0 \leq i < 2^m - \tau : i_{\pi(1)} \neq i_{\pi(1)}, i_{\pi(2)} = j_{\pi(2)}, \ell_i = \ell_j\}$, define i' and j' by (6) and (7) respectively.

$\square$

Proof of Theorem 3: Similar to Theorem 1, Theorem 3 can be proven as follows.

1. For $\mathbf{J}_1(\tau) = \{0 \leq i < 2^m - \tau : i_{\pi(1)} = j_{\pi(1)}, \ell_i = \ell_j\}$, set i' and j' to be two integers defined by (9) and (10) respectively where the integer v is defined as (8). In this case: If $\ell_i \in \{d_0 i_{\pi(2)} + i_{\pi(m)}, i_{\pi(1)} + d_0 i_{\pi(2)} + i_{\pi(m)} : d_0 = 0, 1\}$, then $i_{\pi(m)} = j_{\pi(m)}$ or ($i_{\pi(2)} \neq j_{\pi(2)}$ and $i_{\pi(m)} \neq j_{\pi(m)}$); If $\ell_i \in \{d_0 i_{\pi(2)} + i_{\pi(3)}, i_{\pi(1)} + d_0 i_{\pi(2)} + i_{\pi(3)} : d_0 = 0, 1\}$, then $i_{\pi(3)} = j_{\pi(3)}$ and ($i_{\pi(2)} \neq j_{\pi(2)}$ and $i_{\pi(3)} \neq j_{\pi(3)}$);

2. For $\mathbf{J}_2(\tau) = \{0 \leq i < 2^m - \tau : i_{\pi(1)} \neq i_{\pi(1)}, j_{\pi(2)} = j_{\pi(2)}, \ell_i = \ell_j\}$, set i' and j' to be the two integers defined by (13) and (14) respectively;

3. For $\mathbf{J}_3(\tau) = \{0 \leq i < 2^m - \tau : i_{\pi(1)} \neq i_{\pi(1)}, i_{\pi(2)} \neq j_{\pi(2)}, \ell_i = \ell_j\}$, set s and t to be the two integers defined by (11) and (12) respectively.

$\square$

4 Odd Periodic Complementary Pairs

4.1 Results

Theorem 4 *Let $p = 1$ and $q = 2$. The pair of sequences a and b given by (5) is an odd periodic complementary pair if $\pi(1) = 1$, $c_1 \in \{0, H/2\}$, and $\ell_i \in \{d_0 i_{\pi(1)} + i_{\pi(2)}, d_0 i_{\pi(1)} + i_{\pi(m)} : d_0 = 0, 1\}$.*

Theorem 5 *Let $p = 2$ and $q = 4$. The pair of sequences a and b given by (5) is an odd periodic complementary pair if $\pi(2) = 1$, $c_1 \in \{0, H/2\}$, and $\ell_i \in \{i_{\pi(3)}, i_{\pi(m)}, i_{\pi(1)} + i_{\pi(2)} + i_{\pi(4)}, i_{\pi(1)} + i_{\pi(2)} + i_{\pi(3)} + i_{\pi(4)} + i_{\pi(m)}\}$.*

Define a mapping $\pi'(k) = \pi(m+1-k)$, $k \in \{1, \cdots, m\}$, and replace π by π', the following corollary followes from Theorems 4 and 5.

Corollary 2 *The pair of sequences a and b given by (5) is an odd periodic complementary pair, if*

1. $p = m-1$ *and* $q = m$, $\pi(m) = 1$, $c_1 \in \{0, H/2\}$, *and* $\ell_i \in \{d_0 i_{\pi(m)} + i_{\pi(m-1)}, d_0 i_{\pi(m)} + i_{\pi(1)} : d_0 = 0, 1\}$.

2. $p = m-3$ *and* $q = m-1$, $\pi(m-1) = 1$, $c_1 \in \{0, H/2\}$, *and* $\ell_i \in \{i_{\pi(m-2)}, i_{\pi(1)}, i_{\pi(m)} + i_{\pi(m-1)} + i_{\pi(m-3)}, i_{\pi(m)} + i_{\pi(m-1)} + i_{\pi(m-2)} + i_{\pi(m-3)} + i_{\pi(1)}\}$.

Example 3 *Let* $m = 5$, $H = 2$ *and* π *be the identity permutation. Let* $a_i = \sum_{k=2}^{4} i_k i_{k+1}$ *and* $b_i = a_i + i_5$, $0 \leq i \leq 31$. *The odd periodic autocorrelation function of binary sequences a and b are listed in Table 3. Furthermore they form an odd periodic complementary sequence pair.*

Table 3: The autocorrelation function of binary sequences a and b

a	$(0,0,0,1,0,0,1,0,0,0,0,1,1,1,0,1,0,0,0,1,0,0,1,0,0,0,0,1,1,1,0,1)$
b	$(0,1,0,0,0,1,1,1,0,1,0,0,1,0,0,0,0,1,0,0,0,1,1,1,0,1,0,0,1,0,0,0)$
$\{\hat{R}_a(\tau)\}_{\tau=1}^{31}$	$(2,0,2,0,2,0,2,0,6,0,-10,0,-2,0,-2,0,2,0,2,0,10,$ $0,-6,0,-2,0,-2,0,-2,0,-2)$
$\{\hat{R}_b(\tau)\}_{\tau=1}^{31}$	$(-2,0,-2,0,-2,0,-2,0,-6,0,10,0,2,0,2,0,-2,0,-2,0,-10,$ $0,6,0,2,0,2,0,2,0,2)$

Example 4 *Let* $m = 5$, $H = 4$ *and* π *be the identity permutation. Let* $a_i = 2\sum_{k=2}^{4} i_k i_{k+1} + 2i_4 + i_5$ *and* $b_i = a_i + 2i_5$, $0 \leq i \leq 31$. *The odd periodic autocorrelation function of quaternary sequences a and b are listed in Table 4. Furthermore they form an odd periodic complementary sequence pair.*

Table 4: The autocorrelation function of quaternary sequences a and b

a	$(0,1,2,1,0,1,0,3,0,1,2,1,2,3,2,1,0,1,2,1,0,1,0,3,0,1,2,1,2,3,2,1)$
b	$(0,3,2,3,0,3,0,1,0,3,2,3,2,1,2,3,0,3,2,3,0,3,0,1,0,3,2,3,2,1,2,3)$
$\{\hat{R}_a(\tau)\}_{\tau=1}^{31}$	$(-2\xi,0,2\xi,0,-2\xi,0,2\xi,0,-6\xi,0,-10\xi,0,2\xi,0,-2\xi,0,-2\xi,$ $0,2\xi,0,-10\xi,0,-6\xi,0,2\xi,0,-2\xi,0,2\xi,0,-2\xi,)$
$\{\hat{R}_b(\tau)\}_{\tau=1}^{31}$	$(2\xi,0,-2\xi,0,2\xi,0,-2\xi,0,6\xi,0,10\xi,0,-2\xi,0,2\xi,0,2\xi,$ $0,-2\xi,0,10\xi,0,6\xi,0,-2\xi,0,2\xi,0,-2\xi,0,2\xi,)$

4.2 Proof of Results

In this subsection, we will prove that

$$\hat{R}_a(\tau) + \hat{R}_b(\tau) = 0, \quad 0 < \tau < 2^m$$

for pair (a, b) given in (5) under the conditions listed in the former subsection.

Proof of Theorem 4: Similarly, Theorem 4 can be proven by using following technique.

1. For $\mathbf{J}_1(\tau) = \{0 \le i < 2^m - \tau : i_{\pi(1)} = j_{\pi(1)}, i_{\pi(2)} = j_{\pi(2)}, \ell_i = \ell_j\}$, let i' and j' to be two integers defined by (9) and (10) respectively where the integer $v \ge 3$ is defined as (8);

2. For $\mathbf{J}_2(\tau) = \{0 \le i < 2^m - \tau : i_{\pi(1)} = j_{\pi(1)}, i_{\pi(2)} \ne j_{\pi(2)}, \ell_i = \ell_j\}$, let i' and j' to be the two integers defined by (13) and (14). In this case, $i_{\pi(m)} = j_{\pi(m)}$ if $\ell_i \in \{i_{\pi(1)} + i_{\pi(m)}, i_{\pi(m)}\}$ respectively;

3. For $\mathbf{J}_3(\tau) = \{0 \le i < 2^m - \tau : i_{\pi(1)} \ne i_{\pi(1)}, j_{\pi(2)} = j_{\pi(2)}, \ell_i = \ell_j\}$, let s and t to be the two integers defined by (11) and (12) respectively;

4. For $\mathbf{J}_4(\tau) = \{0 \le i < 2^m - \tau : i_{\pi(1)} \ne i_{\pi(1)}, i_{\pi(2)} \ne j_{\pi(2)}, \ell_i = \ell_j\}$, let s and t to be the two integers defined by (11) and (12) respectively.

Note that $\mathbf{J}_2(\tau)$ and $\mathbf{J}_3(\tau)$ are empty sets if $\ell_i \in \{i_{\pi(1)} + i_{\pi(2)}, i_{\pi(2)}\}$.

□

Proof of Theorem 5: Theorem 5 can be proven similarly by the following cases.

1. For $\mathbf{J}_1(\tau) = \{0 \le i < 2^m - \tau : i_{\pi(1)} = j_{\pi(1)}, i_{\pi(2)} + i_{\pi(4)} + i_{\pi(m)} = j_{\pi(2)} + j_{\pi(4)} + j_{\pi(m)}, \ell_i = \ell_j\}$, define i' and j' by (9) and (10) respectively where the integer $v \ge 3$ is defined as (8). In this case: If $\ell_i \in \{i_{\pi(3)}, i_{\pi(1)} + i_{\pi(2)} + i_{\pi(3)} + i_{\pi(4)} + i_{\pi(m)}\}$, then $i_{\pi(3)} = j_{\pi(3)}$, which indicates that $v \ne 3$; If $\ell_i \in \{i_{\pi(m)}, i_{\pi(1)} + i_{\pi(2)} + i_{\pi(4)}\}$, then $i_{\pi(m)} = j_{\pi(m)}$ and $i_{\pi(2)} + j_{\pi(4)} = i_{\pi(2)} + j_{\pi(4)}$. The latter equality implies that $v \ne 4$;

2. For $\mathbf{J}_2(\tau) = \{0 \le i < 2^m - \tau : i_{\pi(1)} = j_{\pi(1)}, i_{\pi(2)} + i_{\pi(4)} + i_{\pi(m)} \ne j_{\pi(2)} + j_{\pi(4)} + j_{\pi(m)}, \ell_i = \ell_j\}$, define i' and j' by (13) and (14) respectively;

3. For $\mathbf{J}_3(\tau) = \{0 \le i < 2^m - \tau : i_{\pi(1)} \ne j_{\pi(1)}, i_{\pi(2)} = j_{\pi(2)}, i_{\pi(3)} + j_{\pi(3)} = i_{\pi(4)} + j_{\pi(4)}, \ell_i = \ell_j\}$, define i' and j' by (6) and (7) respectively;

4. For $\mathbf{J}_4(\tau) = \{0 \le i < 2^m - \tau : i_{\pi(1)} \ne j_{\pi(1)}, i_{\pi(2)} = j_{\pi(2)}, i_{\pi(3)} = j_{\pi(3)}, i_{\pi(4)} \ne j_{\pi(4)}, \ell_i = \ell_j\}$, define i' and j' as (9) and (10) respectively by setting the integer $v = 4$;

5. For $\mathbf{J}_5(\tau) = \{0 \le i < 2^m - \tau : i_{\pi(1)} \ne j_{\pi(1)}, i_{\pi(2)} = j_{\pi(2)}, i_{\pi(3)} \ne j_{\pi(3)}, i_{\pi(4)} = j_{\pi(4)}, \ell_i = \ell_j\}$, define i' and j' by (13) and (14) respectively. Note that $\mathbf{J}_5(\tau)$ is an empty set if $\ell_i \in \{i_{\pi(3)}, i_{\pi(1)} + i_{\pi(2)} + i_{\pi(4)}\}$; If $\ell_i \in \{i_{\pi(m)}, i_{\pi(1)} + i_{\pi(2)} + i_{\pi(3)} + i_{\pi(4)} + i_{\pi(m)}\}$, then $i_{\pi(m)} = j_{\pi(m)}$;

6. For $\mathbf{J}_6(\tau) = \{0 \le i < 2^m - \tau : i_{\pi(1)} \ne j_{\pi(1)}, i_{\pi(2)} \ne j_{\pi(2)}, i_{\pi(3)} + j_{\pi(3)} = i_{\pi(4)} + j_{\pi(4)}, \ell_i = \ell_j\}$, define s and t by (11) and (12) respectively.

□

5 Conclusion

In this paper, we have shown several constructions of even periodic (odd periodic) complementary sequences pairs over Z_H and 4^q-QAM complementary sequences pairs, which are defined using the generalized boolean functions, where $H \equiv 0 \pmod 4$ and $q \geq 2$ is an arbitrary integer. Those constructed sequences pairs can be used to design optimal signal sets. An interesting problem is to search for perfect even periodic complementary pairs or perfect odd periodic complementary pairs with new length.

References

[1] P.Z. Fan and M. Darnell, *Sequence design for communications applications*, Research Studies Press, John Wiley & Sons Ltd, London, 1996.

[2] J.A. Davis and J. Jedwab, "Peak-to-mean power control in OFDM, Golay complementary sequences and Reed-Muller codes," *IEEE Trans. Inform. Theory*, vol. 45, no. 7, pp. 2397–2417, Nov. 1999.

[3] K.Q. Feng, P. J-S. Shiue, and Q. Xiang, "On aperiodic and periodic complementary binary sequences," *IEEE Trans. Inf. Theory*, vol. 45, no. 1, pp. 296–303, Jan. 1999.

[4] H. Ganapathy, D.A. Pados, and G.N. Karystinos, "New bounds and optimal binary signature sets–Part I: Periodic total squared correlation," *IEEE Trans. Inf. Theory*, vol. 59, no. 4, pp. 1123–1132, 2011.

[5] M.J.E. Golay, "Multislit spectroscopy," *J. Opt. Soc. Amer.*, vol. 39, pp. 437–444, 1949.

[6] M.J.E. Golay, "Complementary series," *IRE Trans. Inform. Theory*, vol. 7, pp. 82–87, 1961.

[7] M.J.E. Golay, "Note on "Complementary series"," *Proc. IRE*, vol. 50, pp. 84, 1962.

[8] S. W. Golomb and G. Gong, *Signal Designs With Good Correlation: For Wireless Communication, Cryptography and Radar Applications.* Cambridge, U.K: Cambridge Univeristy Press, 2005.

[9] H.L. Jin, G.D. Liang, Z.H. Liu, and C.Q. Xu, "The Necessary Condition of Families of Odd periodic Perfect Complementary Sequence Pairs," *2009 International Conference on Computational Intelligence and Security*, pp. 303–307, 2009.

[10] G.N. Karystinos and D.A. Pados, "New bounds on the total squared correlation and optimal design of DS-CDMA binary signature sets, " *IEEE Trans. Communi.*, vol. 51, no. 1, pp. 48–51, Jan. 2003.

[11] N. Levanon, Radar Principles. New York: wiley, 1988.

[12] H.D. Lüke, "Binary odd periodic complementary sequences," *IEEE Trans. Inform. Theory*, vol. 43, no. 1, pp. 365–367, Jan. 1997.

[13] H.D. Lüke and H.D. Schotten, "Odd-perfect almost binary correlation sequences," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 31, pp. 495–498, Jan. 1995.

[14] M.G. Parker, K.G. Paterson, and C. Tellambura, "Golay complementary sequences," *in Wiley Encyclopedia of Telecommunications*, J. G. Proakis, Ed. New York: Wiley Interscience, 2002.

[15] K.G. Paterson, "Generalized Reed-Muller codes and power control for OFDM modulation," *IEEE. Trans. Inform. Theory*, vol. 46, no. 1, pp. 104–120, Feb. 2000.

[16] M.B. Pursley, *A Introduction to Digital Communications.* Pearson Prentice Hall, U.S, 2005.

[17] M.B. Pursley, "Performance evaluation for phase-coded spread-spectrum multiple-access communication—Part I: System analysis," *IEEE Trans. Inform. Theory*, vol. com-25, no. 8, pp. 795-799, Aug. 1977.

[18] M.B. Pursley, "Performance evaluation for phase-coded spread-spectrum multiple-access communication—Part II: Code sequence analysis," *IEEE Trans. Inform. Theory*, vol. com-25, no. 8, pp. 800-803, Aug. 1977.

[19] D.V. Sarwate and M. B. Pursley, "Crosscorrelation properties of pseudorandom and related sequences," *Proc. IEEE*, vol. 68, pp. 593–619, May 1980.

[20] H.D. Schotten, "New optimum ternary complementary sets and almost quadriphase, perfect sequences," *in Int. Conference on Neural Networks and Signal Processing (ICNNSP'95)*, Nanjing, China, Dec. 10-13, 1995, pp. 1106–1109, 1995.

[21] R. Sivaswamy, "Self-clutter cancellation and ambiguity properties of subcomplementary sequences," *IEEE Trans, Aerosp. Electron. Sysr.*, vol. AES-18, pp. 163–180, Mar. 1982.

[22] R.J. Turyn, "Hadamard matrices, Baumert-Hall units, four-symbol sequences, pulse compression, and surface wave encodings," *J. Combin. Theory (A)*, vol. 16, no. 3, pp. 313–333, 1974.

[23] C.C. Tseng and C.L. Liu, "Complementary sets of sequences," *IEEE Trans. Inform. Theory*, vol. 18, pp. 644–651, May 1972.

[24] H. Wen, F. Hu, and F. Jin, "Design of odd periodic complementary binary signal set," *Ninth IEEE Symposium on Computers and Communications 2004*, vol. 2, pp. 590–593, 2004.