Quadratic Zero-Difference Balanced Functions, APN Functions and Strongly Regular Graphs

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Abstract

Let F be a function from $\mathbb{F}_{p^n}$ to itself and δ a positive integer. F is called zero-difference δ-balanced if the equation $F(x+a)-F(x)=0$ has exactly δ solutions for all nonzero $a \in \mathbb{F}_{p^n}$. As a particular case, all known quadratic planar functions are zero-difference 1-balanced; and some quadratic APN functions over $\mathbb{F}_{2^n}$ are zero-difference 2-balanced. In this paper, we study the relationship between this notion and differential uniformity; we show that all quadratic zero-difference δ-balanced functions are differentially δ-uniform and we investigate in particular such functions with the form $F = G(x^d)$, where $\gcd(d, p^n-1) = \delta+1$ and where the restriction of G to the set of all nonzero $(\delta+1)$-th powers in $\mathbb{F}_{p^n}$ is an injection. We introduce new families of zero-difference p^t-balanced functions. More interestingly, we show that the image set of such functions is a regular partial difference set, and hence yields strongly regular graphs; this generalizes the constructions of strongly regular graphs using planar functions by Weng et al. Using recently discovered quadratic APN functions on $\mathbb{F}_{2^s}$, we obtain 15 new $(256, 85, 24, 30)$ negative Latin square type strongly regular graphs.

Keywords: Zero-difference balanced functions, almost perfect nonlinear functions, strongly regular graphs.

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1 Introduction

Functions defined over $\mathbb{F}_{p^n}$ with high nonlinearity have been studied extensively in the last three decades as they are widely used in symmetric cipher design, allowing resisting known attacks. For instance, permutations over $\mathbb{F}_{2^n}$ with high nonlinearity and low differential uniformity (defined in Section 2.1) are chosen as the Substitution boxes in block ciphers to bring the necessary confusion, as shown in [31]. Besides this, they are interesting thanks to their close relationship with notions in coding theory and combinatorics. For instance, almost bent functions can be used to construct codes, schemes, graphs and authentication schemes [10, 14]; bent functions can be used to construct schemes and strongly regular graphs [35, 36]; quadratic almost perfect nonlinear functions (APN, defined in Section 2.1) can be used to construct dual hyperplanes [21]; perfect nonlinear functions (PN, defined in Section 2.1) can be used to construct difference sets and strongly regular graphs [42]. For a survey of highly nonlinear functions, one may refer to [7, 8, 9]. In this paper, we will study some quadratic functions having some peculiarity implying the properties above, and establish a new relationship between them and strongly regular graphs.

In [15], Ding introduced a special kind of functions called *zero-difference balanced* (ZDB) functions to construct codes with good property. A function from an abelian group A to the other abelian group B is called zero-difference δ-balanced if the equation $F(x+a)-F(x)=0$ has exactly δ solutions for all nonzero $a \in A$, where δ is some positive integer. Recently, several classes of zero-difference balanced functions were constructed and more of their applications to the construction of combinatorial objects were explored, see [5, 15, 16, 18, 17, 40, 44] and the references therein. It is worth to mention that the known constructions of ZDB functions are mostly defined on a cyclic group because such ZDB functions have more applications. Throughout this paper, we shall consider ZDB functions on a non-cyclic group (more precisely the additive group of the field $\mathbb{F}_{p^n}$) and show that some of them may be applied to construct partial difference sets. We shall first show that a quadratic zero-difference δ-balanced function is a quadratic differentially δ-uniform function. The converse of the above statement is not true in general: by [24, 42], all quadratic PN functions are zero-difference 1-balanced, up to the addition of a linear function, but this is not true for APN functions on $\mathbb{F}_{2^n}$; for example, 18 of the $2,275$ newly discovered quadratic APN functions on $\mathbb{F}_{2^8}$ in [39, 41] are zero-difference 2-balanced; and when n is odd, APN permutations are clearly not zero-difference 2-balanced. However, the notion of zero-difference balance gives a nice enlightenment on the APNness of some classes of known APN functions (see below).

For the construction of zero-difference δ-balanced functions, it is shown in Corollary 1 that quadratic functions of the form $F(x) = G(x^d)$ satisfy the requirement when $\gcd(d, p^n-1) = \delta+1$ and when the restriction of G to the set of $(\delta+1)$-th powers of $\mathbb{F}_{p^n}$ is an injection. By discovering such G, we obtain new families of differentially δ-uniform functions. In Section 4, on the one hand, we provide new methods to construct zero-difference δ-balanced functions; and on the other hand, new families of zero-difference p-balanced functions are presented. As a particular case, new APN functions are obtained.

It is proven in [42] that, given an even PN function on $\mathbb{F}_{p^n}$ (i.e. $f(0) = 0$ and $f(-x) = f(x)$ for all $x \in \mathbb{F}_{p^n} \setminus \{0\}$), its image set (excluding 0) is either a Payley difference set when $p^n \equiv 3 \mod 4$, or a Payley partial difference set when $p^n \equiv 1 \mod 4$. In Section 5, we establish similar results. Precisely, let F be a zero-difference p^t-balanced function on $\mathbb{F}_{p^n}$, where $n \equiv 0 \mod 2t$ and $t > 0$ (the case $t = 0$ is actually the result obtained in [42]), denoting $D = \mathrm{Im}(F) \setminus \{0\}$, then D is a regular

$$\left(p^n, \frac{p^n-1}{p^t+1}, \frac{p^n - 3p^t - 2 - \epsilon p^{n/2+2t} + \epsilon p^{n/2+t}}{(p^t+1)^2}, \frac{p^n - \epsilon p^{n/2} + \epsilon p^{n/2+t} - p^t}{(p^t+1)^2}\right)$$

partial difference set (PDS, defined in Section 2.3), where $n = 2kt$ and $\epsilon = (-1)^k$. Therefore, we obtain a new construction of strongly regular graphs (SRG, defined in Section 2.3) by its relationship to partial difference sets. Particularly, when k is even, we obtain negative Latin square type SRGs. In Section 6, using newly discovered zero-difference 2-balanced (namely APN) functions on $\mathbb{F}_{2^s}$ and by comparing the SRGs with parameter $(256, 85, 24, 30)$ to known constructions, we found 15 new such graphs.

The rest of the paper is organized as follows. In Section 2, we give necessary definitions and results. The properties of zero-difference balanced functions are presented in Section 3. Section 4 presents constructions of zero-difference p^t-balanced functions. In Section 5, we establish the relationship between zero-difference p^t-balanced functions and partial difference sets (strongly regular graphs), and in Section 6 we discuss the newness of the SRGs obtained from zero-difference 2-balanced functions. Some concluding remarks are given in Section 7.

2 Preliminary

In this section, we introduce basic definitions and results which will be used in the following sections.

2.1 Functions defined over $\mathbb{F}_{p^n}$

Let F be a function from $\mathbb{F}_{p^n}$ to itself. For any $a, b \in \mathbb{F}_{p^n}$; $a \neq 0$, define the difference function $\delta_F(a, b) = |\{x : x \in \mathbb{F}_{p^n} | F(x + a) - F(x) = b\}|$, where $|S|$ denotes the size of a set S. Let $\Delta_F = \max_{a,b \in \mathbb{F}_{p^n}, a \neq 0} \delta_F(a, b)$, the function F is called a *differentially Δ_F-uniform function*. Particularly, when $p = 2$, it is easy to see that the smallest value of Δ_F is 2, we call a function with such value of Δ_F *almost perfect nonlinear* (APN); and when p is odd and $\Delta_F = 1$, we call such functions *perfect nonlinear* (PN) or *planar*. The multiset $\mathcal{D}_F := \{\delta_F(a, b) : a \in \mathbb{F}_{p^n}, b \in \mathbb{F}_{p^n}, a \neq 0\}$ is called the *differential spectrum* of F.

Another commonly used parameter evaluating the nonlinearity of F is as follows. For the above function F, the *Walsh transform* of F is defined as

$$\mathcal{W}_F(a, b) := \sum_{x \in \mathbb{F}_{p^n}} \zeta_p^{\mathrm{Tr(aF(x)+bx)}}, \quad a \in \mathbb{F}_{p^n}^* = \mathbb{F}_{p^n} \setminus \{0\}, b \in \mathbb{F}_{p^n},$$

where ζ_p is a complex primitive p-th root of unity, and $\mathrm{Tr(x)} := \sum_{\mathrm{i}=0}^{\mathrm{n}-1} \mathrm{x}^{\mathrm{p}^{\mathrm{i}}}$ denotes the usual trace function from $\mathbb{F}_{p^n}$ to $\mathbb{F}_p$. The multiset $\mathcal{W}_F := \{\mathcal{W}_F(a,b) : a \in \mathbb{F}_{p^n}^*, b \in \mathbb{F}_{p^n}\}$ is called the *Walsh spectrum* of F, and each number $\mathcal{W}_F(a,b)$ is called the *Walsh coefficient* at (a,b). Particularly, for a p-ary function $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$, the Walsh transform of f equals $\sum_{x\in\mathbb{F}_{p^n}} \zeta_p^{f(x)+\mathrm{Tr(bx)}}$, which is denoted by $\mathcal{W}_f(b)$. When $p = 2$, the *nonlinearity* of F is defined as $\mathrm{NL}(F) = 2^{n-1} - \frac{1}{2}\max_{a,b\in\mathbb{F}_{2^n},a\neq 0} \mathcal{W}_F(a,b)$.

Finally, the *algebraic degree* of a function $F(x) = \sum_{i=0}^{p^n-1} a_i x^i \in \mathbb{F}_{p^n}[x]$, denoted by $\deg F$, is defined as the maximal p-weight of the exponent i such that $a_i \neq 0$, where the p-weight of an integer i is the sum in $\mathbb{Z}$ of the coefficients in its p-ary expression. Particularly, F is called *quadratic* if $\deg F = 2$ (sometimes it is called *Dembowski-Ostrom*, in brief, DO, if $F(x) = \sum_{i,j=0}^{n-1} a_{ij} x^{p^i+p^j}$); and F is called *affine* if $\deg F \leq 1$. Two functions $F_1, F_2 : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ are called *extended affine* (EA)- equivalent if there exist linear permutations L_1, L_2 and an affine function A of $\mathbb{F}_{p^n}$ such that $F_2 = L_1 \circ F_1 \circ L_2 + A$. Furthermore, F_1 and F_2 are called *Carlet-Charpin-Zinoviev*-equivalent, in brief, (CCZ)-equivalent, if there exists an affine permutation ϕ of $\mathbb{F}_{p^n} \times \mathbb{F}_{p^n}$ such that $\phi(\{(x, F_1(x)) : x \in \mathbb{F}_{p^n}\}) = \{(x, F_2(x)) : x \in \mathbb{F}_{p^n}\}$. It is well known that CCZ-equivalence implies EA-equivalence, but not vice versa. Interested readers may refer to [8] for more details.

2.2 Group rings and character theory

Group rings and character theory of finite fields are useful tools to study functions defined on $\mathbb{F}_{p^n}$ and their related combinatorial objects. We briefly review some definitions and results. For more details on group rings and character theory, please refer to [33] and [30] respectively. In the following, we assume $\mathcal{G}$ is a finite Abelian group. The group algebra $\mathbb{C}[\mathcal{G}]$ consists of all formal sums $\sum_{g\in\mathcal{G}} a_g g, a_g \in \mathbb{C}$. We define componentwise addition

$$\sum_{g\in\mathcal{G}} a_g g + \sum_{g\in\mathcal{G}} b_g g = \sum_{g\in\mathcal{G}} (a_g + b_g) g,$$

and multiplication by

$$\sum_{g\in\mathcal{G}} a_g g \cdot \sum_{g\in\mathcal{G}} b_g g = \sum_{g\in\mathcal{G}} \left(\sum_{h\in\mathcal{G}} a_h \cdot b_{gh^{-1}} \right) g.$$

A subset S of $\mathcal{G}$ is identified with the group ring element $\sum_{s\in S} s$ in $\mathbb{C}[\mathcal{G}]$, which is also denoted by S (by abuse of notation). For $A = \sum_{g\in\mathcal{G}} a_g g$ in $\mathbb{C}[\mathcal{G}]$ and t an integer, we define $A^{(t)} := \sum_{g\in\mathcal{G}} a_g g^t$.

A character χ of a finite Abelian group $\mathcal{G}$ is a homomorphism from $\mathcal{G}$ to $\mathbb{C}^*$. A character χ is called *principal* if $\chi(c) = 1$ for all $c \in \mathcal{G}$, otherwise it is called *non-principal*. All characters form a group which is denoted by $\widehat{\mathcal{G}}$. This *character group* $\widehat{\mathcal{G}}$ is isomorphic to $\mathcal{G}$. We denote its unity by χ_0 and the unity of $\mathcal{G}$ by $1_{\mathcal{G}}$. The action

of any character χ is extended to $\mathbb{C}[\mathcal{G}]$ by $\chi(\sum_{g\in\mathcal{G}} a_g g) = \sum_{g\in\mathcal{G}} a_g\chi(g)$. The following well-known Inversion formula is very useful to us.

Lemma 1 (Inversion Formula)**.** *Let* $D = \sum_{g\in\mathcal{G}} a_g g \in \mathbb{C}[\mathcal{G}]$. *Then* $a_g = \frac{1}{|\mathcal{G}|}\sum_{\chi\in\widehat{\mathcal{G}}} \chi(D)\chi(g^{-1})$.

The following lemma is an application of the Inversion formula.

Lemma 2. *Let* $D_1, D_2 \in \mathbb{C}[\mathcal{G}]$ *be two group ring elements. Then* $D_1 = D_2$ *if and only if* $\chi(D_1) = \chi(D_2)$ *for all characters of* $\mathcal{G}$.

Finally, for the finite field $\mathbb{F}_{p^n}$, define $\chi_1 : \mathbb{F}_{p^n} \to \mathbb{C}$ as $\chi_1(x) := \zeta_p^{\mathrm{Tr(x)}}$ for all $x \in \mathbb{F}_{p^n}$. Then χ_1 is an additive character of $\mathbb{F}_{p^n}$. Moreover, every additive character χ is of the form χ_b ($b \in \mathbb{F}_{p^n}$), where χ_b is defined by $\chi_b(x) = \chi_1(bx)$ for all $x \in \mathbb{F}_{p^n}$.

2.3 Partial difference sets and strongly regular graphs

Let $\mathcal{G}$ be a multiplicative group of order v. A k-subset D of $\mathcal{G}$ is called a (v, k, λ, μ) *partial difference set* (PDS) if each non-identity element in D can be represented as gh^{-1} $(g, h \in D, g \neq h)$ in exactly λ ways, and each non-identity element in $\mathcal{G}\backslash D$ can be represented as gh^{-1} $(g, h \in D, g \neq h)$ in exactly μ ways. We shall always assume that the identity element $1_\mathcal{G}$ of $\mathcal{G}$ is not contained in D. Particularly, D is called *regular* if, denoting $D^{(-1)} := \{d^{-1};\, d \in D\}$, we have $D^{(-1)} = D$. Using the group ring language, a k-subset D of $\mathcal{G}$ with $1_\mathcal{G} \notin D$ is a (v, k, λ, μ)-PDS if and only if the following equation holds:

$$DD^{(-1)} = (k-\mu)1_\mathcal{G} + (\lambda-\mu)D + \mu\mathcal{G}. \tag{1}$$

Particularly, for a PDS, when $\lambda = \mu$, this reduces to the so-called difference set. A k-subset D of $\mathcal{G}$ is called a (v, k, λ) *difference set* (DS) if each nonidentity element of $\mathcal{G}$ can be represented in the form $d_1d_2^{-1}$ $(d_1, d_2 \in D, d_1 \neq d_2)$ in exactly λ ways. Similarly, using group ring notation, the subset D is a (v, k, λ) difference set if and only if

$$DD^{(-1)} = k1_\mathcal{G} + \lambda(\mathcal{G} - 1_\mathcal{G}).$$

By Lemma 2, we have the following result to show a k-subset D is a PDS. This result can be found in [32].

Lemma 3 ([32])**.** *Let* D *be a group ring element of* $\mathbb{C}[\mathcal{G}]$ *with* $|D| = k$. *Then*

(i) D *is a* (v, k, λ) *difference set if and only if* $\chi(DD^{(-1)}) = k - \lambda$ *for all non-principal character* χ *and* $k^2 = (k-\lambda) + \lambda v$.

(ii) D *is a* (v, k, λ, μ) *partial difference set if and only if, for any nonprincipal character* χ *of* $\mathcal{G}$,

$$\chi(DD^{(-1)}) = k - \mu + (\lambda-\mu)\chi(D) \tag{2}$$

and $k^2 = (k-\mu) + k(\lambda-\mu) + \mu v$.
If D *is regular then* $\chi(D)^2 = \chi(DD^{(-1)})$ *and the former condition is equivalent to:*

$$m\chi(D) = \frac{(\lambda-\mu) \pm \sqrt{(\mu-\lambda)^2 - 4(\mu-k)}}{2}.$$

Combinatorial objects associated with partial difference sets are strongly regular graphs. A graph Γ with v vertices is called a (v, k, λ, μ) *strongly regular graph* (SRG) if each vertex is adjacent to exactly k other vertices, any two adjacent vertices have exactly λ common neighbours, and any two non-adjacent vertices have exactly μ common neighbours.

Given a group $\mathcal{G}$ of order v and a k-subset D of $\mathcal{G}$ with $1_{\mathcal{G}} \notin D$ and $D^{(-1)} = D$, the graph $\Gamma = (V, E)$ defined as follows is called the *Cayley graph* generated by D in $\mathcal{G}$:

(1) The vertex set V is $\mathcal{G}$;

(2) Two vertices g, h are joined by an edge if and only if $gh^{-1} \in D$.

The following lemma points out the relationship between SRGs and PDSs.

Lemma 4 ([32]). *Let Γ be the Cayley graph generated by a k-subset D of a multiplicative group $\mathcal{G}$ with order v. Then Γ is a (v, k, λ, μ) strongly regular graph if and only if D is a (v, k, λ, μ)-PDS with $1_{\mathcal{G}} \notin D$ and $D^{(-1)} = D$.*

Strongly regular graphs (or partial difference sets) with parameters $(n^2, r(n+\varepsilon), -\varepsilon n + r^2 + 3\varepsilon r, r^2 + \varepsilon r)$ are called of *Latin Square type* if $\varepsilon = -1$, and of *negative Latin Square type* if $\varepsilon = 1$. There are many constructions of SRGs of Latin square type (see Lemma 12 in Section 6), but only a few constructions of negative Latin square type are known. We will show that such graphs may be obtained from quadratic zero-difference δ-balanced functions.

3 A new approach for constructing differentially uniform quadratic functions

In this section, we first discuss the properties of differentially δ-vanishing (defined below) functions, and then use them to construct differentially δ-uniform functions.

3.1 A new notion related to differential uniformity, and its properties

It is well-known that if a function $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ is quadratic, that is, if all of its derivatives $F(x+a) - F(x)$ are affine, then it is differentially δ-uniform if and only if, for every $a \neq 0$ in $\mathbb{F}_{p^n}$, the related homogeneous linear equation $F(x+a) - F(x) = F(a) - F(0)$ has at most δ solutions. We have then:

Proposition 1. *Let $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ be a quadratic function. Then F is differentially δ-uniform if and only if, for every $a \neq 0$, there exists $b \in \mathbb{F}_{p^n}$ such that the equation $F(x+a) - F(x) = b$ has at least one and at most δ solutions in $\mathbb{F}_{p^n}$.*

Indeed, the condition is clearly necessary (take $b = F(a) - F(0)$), and it is also sufficient since the two linear equations $F(x+a) - F(x) = F(a) - F(0)$ and $F(x+a) - F(x) = b$ have then the same number of solutions, since we know that they have both solutions and that they have the same linear part.

An interesting case of application is when b can be taken equal to 0 for every $a \neq 0$.

Definition 1. *Let δ be some positive integer. A function F is called* differentially δ-vanishing *if for every $a \neq 0$, the equation $F(x+a) - F(x) = 0$ has at least one and at most δ solutions in $\mathbb{F}_{p^n}$; it is called* zero-difference δ-balanced *if the equation has δ solutions for every $a \neq 0$ in $\mathbb{F}_{p^n}$.*

The notion of differentially δ-vanishing function is new; the closely related notion of zero-difference δ-balanced function has been introduced in [17].

Remark 1. *Of course, zero-difference δ-balanced implies differentially δ-vanishing.*

For $\delta = 1$ (in odd characteristic) and $\delta = 2$ in characteristic 2, the two notions "zero-difference δ-balanced" and "differentially δ-vanishing" coincide.

According to Proposition 1, we have:

Proposition 2. *Any quadratic differentially δ-vanishing function is differentially δ-uniform.*

We shall see that many known quadratic differentially δ-uniform (in particular, PN and APN) functions, are in fact zero-difference δ-balanced or differentially δ-vanishing (with $\delta = 1$ and 2) and that the notion of zero-difference δ-balanced and differentially vanishing gives a simpler reason why these functions are differentially δ-uniform. It will also lead to new constructions.

We give now a construction of functions F which are differentially vanishing, and more precisely zero-difference balanced.

Proposition 3. *Let d be any positive integer and $e = gcd(d, p^n - 1)$. The function x^d on $\mathbb{F}_{p^n}$ is zero-difference $(e-1)$-balanced.*

Proof. For every $x \in \mathbb{F}_{p^n}$, we have $(x+a)^d = x^d$ if and only if $(x+a)^e = x^e$, which is equivalent to $x + a = wx$ for some e-th root of unity w in $\mathbb{F}_{p^n}$. There are e such w. One of them is $w = 1$; the equation $x + a = wx$ is then impossible. Any other e-th root of unity w gives one distinct solution $x = \frac{a}{w-1}$. □

Of course, composing a differentially δ-vanishing function F on the left by a function which is injective on the image set $\{F(x), x \in \mathbb{F}_{p^n}\}$ of F gives a differentially δ-vanishing function.

Corollary 1. *Let d be any positive integer and G a function from $\mathbb{F}_{p^n}$ to $\mathbb{F}_{p^n}$ such that $G(x^d)$ is quadratic. Let $e = gcd(d, p^n - 1)$ and $C_d = \{x^d : x \in \mathbb{F}_{p^n}\} = C_e$. Then function $F(x) = G(x^d)$ is a differentially $(e-1)$-uniform function if and only if the restriction $G\,|_{C_d}$ of G to C_d is an injection.*

Example 1. *As an application, we revisit the APNness of the well-known APN function $F(x) = x^3 + \mathrm{Tr}(x^9)$ over $\mathbb{F}_{2^n}$, in the case where n is even. We have $F(x) = G(x^3)$, where $G(x) = x + \mathrm{Tr}(x^3)$. Function G is a permutation polynomial. Indeed, if $G(x) = G(x+a)$ for some nonzero a, we get $a + \mathrm{Tr}(a^3 + a^2x + ax^2) = 0$. Since $a \neq 0$ and $a \in \mathbb{F}_2$, then $a = 1$, and therefore $a + \mathrm{Tr}(a^3 + a^2x + ax^2) = 1$, a contradiction. Therefore, G is a permutation and then F is an APN function.*

Several remarks on Corollary 1 can be made:

Remark 2. *(1) When $d = 2$ and p is odd, if the function $F(x) = G(x^2)$ is quadratic and if $G\mid_{C_2}$ is an injection, then F is a perfect nonlinear function. It is further proven in [24, 42] that all quadratic PNs are EA-equivalent to functions of the form $G(x^2)$ where $G\mid_{C_2}$ is an injection.*

(2) When $d = 3$, the function $F(x) = G(x^3)$, if it is quadratic and if $G\mid_{C_3}$ is an injection, is an APN function. The condition "quadratic" seems necessary; there are many examples of permutations G such that $G(x^3)$, non-quadratic, is not APN; for instance there are many examples of power functions x^d over $\mathbb{F}_{2^n}$, n even, where d is co-prime with $2^n - 1$ and such that x^{3d} is not APN.

(3) However, there are also examples of non-quadratic APN functions which are zero-difference 2-balanced. This is the case of all APN power functions over $\mathbb{F}_{2^n}$, n even, since Dobbertin has shown that for these functions x^d, we have $gcd(d, 2^n-1) = 3$ (see a recall of his proof in [8]). The Kasami functions $x^{2^{2k}-2^k+1}$, where k is co-prime with n and the Dobbertin function $x^{2^{\frac{4n}{5}}+2^{\frac{3n}{5}}+2^{\frac{2n}{5}}+2^{\frac{n}{5}}-1}$ where n is divisible by 10 are such functions.

(4) There are many quadratic APN functions which are not of the form $G(x^3)$ (and which are not differentially 2-vanishing); for instance, by checking the $2,275$ quadratic APN functions listed in [39] and [41], we found that only 18 of them are of the form $G(x^3)$. Note that every function of the form $G(x) = \sum_i a_i x^{\frac{2^{k_i}+2^{l_i}}{3}}$ where, for every i, k_i is an odd number and l_i is an even number (and then $\frac{2^{k_i}+2^{l_i}}{3}$ is an integer), is such that $G(x^3)$ is quadratic.

3.2 Further properties

In Corollary 1, function G does not need to be bijective; it just needs to be injective on C_d. But given such G, we can always find a permutation G' on $\mathbb{F}_{p^n}$ such that $G'\mid_{C_d} = G\mid_{C_d}$ and therefore $F(x) = G'(x^d)$. Indeed, any function coinciding with G on C_d and mapping injectively the complement of C_d onto the complement of $G(C_d)$ can be taken for G'. In more precise setting:

Proposition 4. *Let d be a positive integer, $e = gcd(d, p^n - 1)$ and G a function defined on $\mathbb{F}_{p^n}$ such that $G\mid_{C_d}$ is an injection. Let $h : \mathbb{F}_{p^n} \to \mathbb{F}_p$ be the characteristic function of C_d:*

$$h(x) = 1 - \left(x^{\frac{2(p^n-1)}{e}} - x^{\frac{p^n-1}{e}}\right)^{p^n-1},$$

(satisfying $h(x) = 1$ if $x \in C_d = C_e$, and $h(x) = 0$ otherwise). There exists a function $T(x)$ on $\mathbb{F}_{p^n}$ such that the function G' defined by

$$G'(x) = h(x)G(x) + (1 - h(x))T(x) \tag{3}$$

is a permutation and satisfies $G'\mid_{C_d} = G\mid_{C_d}$, that is, $G(x^d) = G'(x^d)$, for all x.

Proof. First, we show that h is indeed the characteristic function of C_d: if $x \in C_d$, that is, $x \in C_e$, then $x^{\frac{p^n-1}{e}} \in \{0,1\}$ implies $x^{\frac{2(p^n-1)}{e}} - x^{\frac{p^n-1}{e}} = 0$ and then $h(x) = 1$; and if $x \notin C_d$, that is, $x \notin C_e$, then $x^{\frac{p^n-1}{e}} \notin \{0,1\}$ implies $x^{\frac{2(p^n-1)}{e}} - x^{\frac{p^n-1}{e}} \neq 0$ and then $h(x) = 0$. Next, clearly there exists a function $T : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ whose restriction $T|_{\mathbb{F}_{p^n} \setminus C_d}$ is a bijection from $\mathbb{F}_{p^n} \setminus C_d$ to $\mathbb{F}_{p^n} \setminus \mathrm{Im}(G|_{C_d})$, then G' defined in (3) is a permutation. Indeed, this is clear from $G'(x) = G(x)$ for $x \in C_d$ and $G'(x) = T(x)$ for $x \notin C_d$, and the properties of G and T. This completes the proof. □

A natural question one may ask is the following:

Problem 1. *Do all differentially δ-vanishing quadratic functions have the form $G(x^d)$ with $\delta = gcd(d, p^n - 1) - 1$ and where $G\,|_{C_d}$ is an injection?*

We leave this problem open.

In the following we characterize the zero-difference balanced functions by their Walsh transform.

Proposition 5. *Let F be a function on $\mathbb{F}_{p^n}$. Then F is zero-difference δ-balanced if and only if*

$$\sum_{b \in \mathbb{F}_{p^n}} \mathcal{W}_F(a,b)\overline{\mathcal{W}_F(a,b)} = \begin{cases} p^{2n} - \delta p^n, & \text{if } a \neq 0, \\ (\delta+1)p^{2n} - \delta p^n, & \text{if } a = 0, \end{cases} \tag{4}$$

where $\overline{w}$ denotes the complex conjugate of the complex number w. If F is differentially δ-uniform, then the condition when $a = 0$:

$$\sum_{b \in \mathbb{F}_{p^n}} \mathcal{W}_F(0,b)\overline{\mathcal{W}_F(0,b)} = (\delta+1)p^{2n} - \delta p^n \tag{5}$$

is sufficient (and so is necessary and sufficient).

Proof. Function F is zero-difference balanced if and only if the numerical function

$$\sigma : u \mapsto |\{x : x \in \mathbb{F}_{p^n} | F(x+u) - F(x) = 0\}|$$

takes constant value δ at every $u \neq 0$ and takes value p^n at 0. This is equivalent to the fact that the numerical function

$$\sigma' : u \mapsto \sum_{b,x \in \mathbb{F}_{p^n}} \zeta_p^{\mathrm{Tr(b(F(x+u)-F(x)))}}$$

takes constant value δp^n at every $u \neq 0$ and takes value p^{2n} at 0. By the properties of the Fourier transform (see e.g. [7]), this is equivalent to the fact that the Fourier transform of σ', that is,

$$\begin{aligned} \widehat{\sigma'}(a) &= \sum_{u \in \mathbb{F}_{p^n}} \sigma'(u)\zeta_p^{\mathrm{Tr(au)}} = \sum_{b,x,u \in \mathbb{F}_{p^n}} \zeta_p^{\mathrm{Tr(b(F(x+u)-F(x)))+Tr(au)}} \\ &= \sum_{b,x,y \in \mathbb{F}_{p^n}} \zeta_p^{\mathrm{Tr(b(F(y)-F(x)))+Tr(a(y-x))}} = \sum_{b \in \mathbb{F}_{p^n}} \mathcal{W}_F(a,b)\overline{\mathcal{W}_F(a,b)} \end{aligned}$$

takes constant value $p^{2n} - \delta p^n$ at every $a \neq 0$ and takes value $\delta p^n(p^n - 1) + p^{2n} = (\delta + 1)p^{2n} - \delta p^n$ at 0. Therefore, F is differentially δ-uniform if and only if (4) holds. Furthermore, if F is differentially δ-uniform, then we know that the number of solutions of $F(x+u) - F(x) = 0$ is not larger than δ. Hence, we need only to show that it is never strictly smaller than δ and this is characterized by $\widehat{\sigma'}(0) = \delta p^n(p^n - 1) + p^{2n}$, that is by (5). □

4 New families of quadratic zero-difference balanced functions over $\mathbb{F}_{p^n}$

In this section, we present new families of quadratic zero-difference balanced functions. In Section 6 below, we will demonstrate that some of these functions give rise to new negative Latin square type strongly regular graphs. As mentioned in the Introduction, the classes of ZDB functions constructed in [15, 40, 5, 18, 44] are on a cyclic group except the one in [18, Theorem 1]. However, one can easily see our constructions below are different from it by comparing the parameters of the ZDB functions. Therefore, the ZDB functions presented in this section are new.

4.1 A class of quadratic zero-difference 2-balanced functions over $\mathbb{F}_{2^n}$

Recall that quadratic zero-difference 2-balanced functions over $\mathbb{F}_{2^n}$ are APN functions. In this section we use Corollary 1 to characterize a family of APN functions of the form $x^3 + \sum_{i=1}^{\ell} \alpha_i \mathrm{Tr}(\beta_i x^3 + \gamma_i x^9)$ defined on $\mathbb{F}_{2^n}$ with n even. We begin with the subclass of those functions of the form $x^3 + \alpha \mathrm{Tr}(\beta x^3 + \gamma x^9)$. When $n = 8$, by choosing proper α, β, γ, it generalizes a sporadic example discovered in [20].

Proposition 6. *Let G be a function on $\mathbb{F}_{2^n}$ defined by*

$$G(x) = x + \alpha \mathrm{Tr}(\beta x + \gamma x^3),$$

where α, β, γ are elements in $\mathbb{F}_{2^n}$, $\alpha \neq 0$ and n is an even integer. Then G is a permutation polynomial of $\mathbb{F}_{2^n}$ if and only if (i) $\gamma = 0$ and $\mathrm{Tr}(\beta\alpha) = 0$, or (ii) $\gamma\alpha^3 = 1$ and $\mathrm{Tr}(\beta\alpha) = 0$. If one of these two conditions is satisfied, the function $F(x) = G(x^3) = x^3 + \alpha \mathrm{Tr}(\beta x^3 + \gamma x^9)$ is a quadratic APN function.

Proof. Function G is a permutation polynomial (PP) if and only if, for every $a \in \mathbb{F}_{2^n}^*$, the equation $G(x + a) + G(x) = 0$ has no solution. This equation is equivalent to:

$$a\alpha^{-1} + \mathrm{Tr}\left(\beta a + \gamma a^3 + (\gamma a^2 + (\gamma a)^{2^{n-1}})x\right) = 0. \tag{6}$$

Since $a \neq 0$, the above equation holds only if $a = \alpha$ and

$$1 + \mathrm{Tr}\left(\beta\alpha + \gamma\alpha^3 + (\gamma\alpha^2 + (\gamma\alpha)^{2^{n-1}})x\right) = 0. \tag{7}$$

If $\gamma\alpha^2+(\gamma\alpha)^{2^{n-1}} \neq 0$, then it is clear that (7) always has solutions. If $\gamma\alpha^2+(\gamma\alpha)^{2^{n-1}} = 0$, that is, if $\gamma = 0$ or $\gamma\alpha^3 = 1$, then (7) has solutions if and only if $\mathrm{Tr}(\beta\alpha + \gamma\alpha^3) = 1$, that is, $\mathrm{Tr}(\beta\alpha) = 1$. This completes the proof. □

Remark 3. *Some remarks on the CCZ-inequivalent APN functions generated by Proposition 6:*

(1) For $\gamma = 0$ and $\mathrm{Tr}(\beta\alpha) = 0$, $G(x)$ is a linear permutation, and hence the APN function $F(x) = G(x^3)$ is CCZ-equivalent to the Gold APN function x^3.

(2) For $\beta = 0$ and $\gamma\alpha^3 = 1$, we have $F(x) = \alpha\left(\frac{x^3}{\alpha} + \mathrm{Tr}\left(\left(\frac{x^3}{\alpha}\right)^3\right)\right)$. On $\mathbb{F}_{2^8}$ and $\mathbb{F}_{2^{10}}$, we checked that taking for α a primitive element of $\mathbb{F}_{2^2}$ gives a function CCZ-inequivalent to $x^3 + \mathrm{Tr}(x^9)$ and x^3.

(3) By exhaustive search of all α, β, γ on $\mathbb{F}_{2^8}$, there are only three (up to equivalence) aforementioned APN functions found. Furthermore, on $\mathbb{F}_{2^{10}}$, by partly search α, β, γ, we get three APN functions, $x^3, x^3+\mathrm{Tr}(x^9)$ and one which is not in any known infinite families.

4.2 Obtaining APN functions from known ones

We give now a method to generate APN functions obtained by Corollary 1 from known ones.

Lemma 5. *Let $G : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ be a function satisfying that $G|_{C_3}$ is an injection, and $h : \mathbb{F}_{2^n} \to \mathbb{F}_2$ be a Boolean function. Let $\gamma \in \mathbb{F}_{2^n}$ be a nonzero constant. Then the function $H : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ defined by $H(x) = G(x) + \gamma h(x)$ has also injective restriction to C_3 if and only if*

$$h(x^3) + h(y^3) = 0 \quad \text{holds for any } x, y \text{ satisfying } G(x^3) + G(y^3) = \gamma. \tag{8}$$

The set $S_{G,\gamma} := \{h \in \mathcal{BF}_n | G(x)+\gamma h(x) \text{ is an injection on } C_3\}$ is a subspace of $(\mathcal{BF}_n, +)$, where $\mathcal{BF}_n$ is the set of all Boolean functions.

Proof. First, we assume that condition (8) holds and we show that $H\mid_{C_3}$ is an injection, that is, $H(x^3)+H(y^3) = 0$ if and only if $x^3 = y^3$. Expanding H we get $G(x^3)+G(y^3) = \gamma(h(x^3) + h(y^3))$. If $h(x^3) + h(y^3) = 0$, we have $G(x^3) + G(y^3) = 0$ and hence $x^3 = y^3$ by the injectivity of G. Otherwise, $G(x^3) + G(y^3) = \gamma$ when $h(x^3) + h(y^3) = 1$, but this is not possible by condition (8). Conversely, assume that $H\mid_{C_3}$ is an injection and there exist x_0, y_0 such that $G(x_0^3) + G(y_0^3) = \gamma$ and $h(x_0^3) + h(y_0^3) = 1$, this would lead to $H(x_0^3) + H(y_0^3) = 0$ and $x_0^3 \neq y_0^3$, which is not possible as $H\mid_{C_3}$ is an injection.
Finally, it is easy to see that, for $h_1, h_2 \in S_{G,\gamma}$, $h_1 + h_2$ satisfies condition (8) and hence $h_1 + h_2 \in S_{G,\gamma}$, which implies that $S_{G,\gamma}$ is a subspace. □

Lemma 5 above provides an algorithmic method to find functions $H = G + \gamma h$ with the property that $H\mid_{C_3}$ is an injection from known such functions G by solving a linear

system of equations in h: we regard h as the vector of length 2^n in the last column of its truth table (by abuse of notation, we still denote this vector by h); a Boolean function h belongs then to the vector space $S_{G,\gamma}$ if and only if $R \times h^T = 0$, where $\times$ denotes the matrix product and R is the matrix whose term at row indexed by an ordered pair $\{x^3, y^3\}$ such that $G(x^3) + G(y^3) = \gamma$ and at column indexed by $u \in \mathbb{F}_{2^n}$ equals 1 if $u = x^3$ or $u = y^3$ and equals 0 otherwise.

If $G(x^3)$ is quadratic, then $H(x^3)$ will be quadratic if and only if $h(x^3)$ is quadratic. As already observed, this is achieved when the degree 2 part of $h(x)$ has the form $\sum_i a_i x^{\frac{2^{k_i}+2^{l_i}}{3}}$ where, for every i, k_i is an odd positive number and l_i is an even positive number, and where $1 < \frac{2^{k_i}+2^{l_i}}{3} < 2^n - 1$ (since $\frac{2^{k_i}+2^{l_i}}{3} = 2^n - 1$ is impossible, unless $n \leq 2$). Note that such h is Boolean when $a_j = a_i^2$ for every i, j such that $l_j = k_i + 1$ and $k_j = l_i + 1$, and for every i, j such that $k_i = l_j + n - 1$ and $l_i = k_j + n - 1$.

4.3 Zero-difference p-balanced functions over $\mathbb{F}_{p^n}$

We present quadratic zero-difference p-balanced functions over $\mathbb{F}_{p^n}$ in this section. Note that such functions are differentially p-uniform functions by Remark 1 and Proposition 2. The following well-known lemma will be useful for the construction. We give a proof for self-completeness.

Lemma 6. *Let $t \in \mathbb{F}_{p^n}$ with n even, the equation $x + x^p = t$ has solutions on $\mathbb{F}_{p^n}$ if and only if $\sum_{\substack{i=0 \\ i \text{ even}}}^{n-2} t^{p^i} \in \mathbb{F}_p$. The number of solutions is then p.*

Proof. If the equation $x + x^p = t$ has a solution $x \in \mathbb{F}_{p^n}$, then we have $\mathrm{Tr(x)} = \sum_{\substack{\mathrm{i=0} \\ \mathrm{i\ even}}}^{\mathrm{n-2}} \mathrm{t^{p^i}} \in \mathbb{F}_\mathrm{p}$. All solutions of the equation are then $x + i$, $i \in \mathbb{F}_p$.

Conversely, assume that $\sum_{\substack{i=0 \\ i \text{ even}}}^{n-2} t^{p^i} \in \mathbb{F}_p$. Let β be a solution of the equation $x + x^p = t$ in some extension field of $\mathbb{F}_{p^n}$. Then:

$$\begin{aligned}
\beta^{p^n} - \beta &= (\beta^{p^n} + \beta^{p^{n-1}}) - (\beta^{p^{n-1}} + \beta^{p^{n-2}}) + \ldots + (\beta^{p^2} + \beta^p) - (\beta^p + \beta) \\
&= (\beta^p + \beta)^{p^{n-1}} - (\beta^p + \beta)^{p^{n-2}} + \ldots + (\beta^p + \beta)^p - (\beta^p + \beta) \\
&= -\sum_{\substack{i=0 \\ i \text{ even}}}^{n-2} t^{p^i} + \left(\sum_{\substack{i=0 \\ i \text{ even}}}^{n-2} t^{p^i} \right)^p = 0.
\end{aligned}$$

Therefore we get $\beta \in \mathbb{F}_{p^n}$ and this completes the proof. □

Now we are ready to present the main result of this section, which is an extension of the results of Section 4.1 to general characteristic.

Theorem 1. *Let $F(x) = x^{p+1} + \alpha \mathrm{Tr}(\beta \mathrm{x}^{\mathrm{p}+1} + \gamma \mathrm{x}^{\mathrm{p}^3+1})$ defined on $\mathbb{F}_{p^n}$, where $\alpha, \beta, \gamma \in \mathbb{F}_{p^n}$. Then*

(i) when $n = 4$, function F is a zero-difference p-balanced function if and only if $\mathrm{Tr}(\alpha\beta + \gamma\alpha^{\mathrm{p}^3}) \neq -1$.

(ii) when $n = 6$, if $\gamma^{p^3-1} = -1$ and $-1 - \mathrm{Tr}(\alpha\beta) \neq 0$, then function F is a zero-difference p-balanced function.

Proof. To prove that F is a zero-difference p-balanced, we need to show that, for any nonzero $a \in \mathbb{F}_{p^n}$, the equation $\Delta_F(ax) = F(ax + a) - F(ax) = 0$ has exactly p solutions. Expanding $\Delta_F(ax)$ we have

$$-a^{p+1}(1 + x + x^p) = \alpha \mathrm{Tr}\left(\beta \mathrm{a}^{\mathrm{p}+1}(1 + \mathrm{x} + \mathrm{x}^{\mathrm{p}}) + \gamma \mathrm{a}^{\mathrm{p}^3+1}(1 + \mathrm{x} + \mathrm{x}^{\mathrm{p}^3})\right). \tag{9}$$

Clearly the above equation holds if and only if, for some $k \in \mathbb{F}_p$, we have:

$$\begin{cases} 1 + x + x^p = ka^{-p-1}\alpha \\ -k = \mathrm{Tr}\left(\beta \mathrm{a}^{\mathrm{p}+1}(1 + \mathrm{x} + \mathrm{x}^{\mathrm{p}}) + \gamma \mathrm{a}^{\mathrm{p}^3+1}(1 + \mathrm{x} + \mathrm{x}^{\mathrm{p}^3})\right) \end{cases}.$$

In the case that $k = 0$, corresponding to $x + x^p = -1$, we have $1 + x + x^{p^3} = \left((1 + x + x^p) - (1 + x^p + x^{p^2}) + (1 + x^{p^2} + x^{p^3})\right) = 0$, and then the second equation in the system above gives $0 = 0$, which implies that all solutions of $x + x^p = -1$ are the solutions of (9). Note also that the equation $x + x^p = -1$ has p solutions in $\mathbb{F}_{p^4}$ (resp. $\mathbb{F}_{p^6}$) by Lemma 6. Therefore, F is zero-difference p-balanced if and only if, $\alpha = 0$ or, for any $k \neq 0$, the system above has no solution. The second equation in the system is equivalent to the following equation, obtained by replacing $1 + x + x^p$ by its value from the first equation, using that $\mathrm{Tr}(\mathrm{ku}) = \mathrm{kTr}(\mathrm{u})$ for every $u \in \mathbb{F}_{p^4}$ (resp. $\mathbb{F}_{p^6}$), and dividing by k:

$$-1 - \mathrm{Tr}(\alpha\beta) = \mathrm{Tr}\left(\gamma(\mathrm{a}^{\mathrm{p}^3-\mathrm{p}}\alpha - \mathrm{a}^{\mathrm{p}^3+1-\mathrm{p}^2-\mathrm{p}}\alpha^{\mathrm{p}} + \mathrm{a}^{1-\mathrm{p}^2}\alpha^{\mathrm{p}^2})\right). \tag{10}$$

(i) In the case $n = 4$, by Lemma 6, the equation $x + x^p = ka^{-p-1}\alpha - 1$ has solutions in $\mathbb{F}_{p^4}$ if and only if $\ell = a^{-p-1}\alpha + a^{-p^3-p^2}\alpha^{p^2}$ belongs to $\mathbb{F}_p$, and therefore also equals $a^{-p^2-p}\alpha^p + a^{-1-p^3}\alpha^{p^3}$. Now, considering the right hand side of (10), we have

$$\begin{aligned}
\mathrm{RHS} &= \mathrm{Tr}\left(\gamma \mathrm{a}^{\mathrm{p}^3-\mathrm{p}}\alpha + \gamma \mathrm{a}^{1-\mathrm{p}^2}\alpha^{\mathrm{p}^2} - \gamma\alpha^{\mathrm{p}}\mathrm{a}^{\mathrm{p}^3+1-\mathrm{p}^2-\mathrm{p}}\right) \\
&= \mathrm{Tr}\left(\gamma \mathrm{a}^{\mathrm{p}^3-\mathrm{p}}\alpha + \gamma \mathrm{a}^{1-\mathrm{p}^2}\alpha^{\mathrm{p}^2} - \gamma \mathrm{a}^{\mathrm{p}^3+1}(\alpha^{\mathrm{p}}\mathrm{a}^{-\mathrm{p}^2-\mathrm{p}})\right) \\
&= \mathrm{Tr}\left(\gamma \mathrm{a}^{\mathrm{p}^3-\mathrm{p}}\alpha + \gamma \mathrm{a}^{1-\mathrm{p}^2}\alpha^{\mathrm{p}^2} - \gamma \mathrm{a}^{\mathrm{p}^3+1}(\ell - \alpha^{\mathrm{p}^3}\mathrm{a}^{-1-\mathrm{p}^3})\right) \\
&= \mathrm{Tr}\left(\gamma \mathrm{a}^{\mathrm{p}^3-\mathrm{p}}\alpha + \gamma \mathrm{a}^{1-\mathrm{p}^2}\alpha^{\mathrm{p}^2} - \ell\gamma \mathrm{a}^{\mathrm{p}^3+1} + \gamma\alpha^{\mathrm{p}^3}\right) \\
&= \mathrm{Tr}\left(\gamma \mathrm{a}^{\mathrm{p}^3-\mathrm{p}}\alpha + \gamma \mathrm{a}^{1-\mathrm{p}^2}\alpha^{\mathrm{p}^2} - (\mathrm{a}^{-\mathrm{p}-1}\alpha + \mathrm{a}^{-\mathrm{p}^3-\mathrm{p}^2}\alpha^{\mathrm{p}^2})\gamma \mathrm{a}^{\mathrm{p}^3+1} + \gamma\alpha^{\mathrm{p}^3}\right) \\
&= \mathrm{Tr}\left(\gamma \mathrm{a}^{\mathrm{p}^3-\mathrm{p}}\alpha + \gamma \mathrm{a}^{1-\mathrm{p}^2}\alpha^{\mathrm{p}^2} - (\gamma\alpha \mathrm{a}^{\mathrm{p}^3-\mathrm{p}} + \gamma\alpha^{\mathrm{p}^2}\mathrm{a}^{1-\mathrm{p}^2}) + \gamma\alpha^{\mathrm{p}^3}\right) = \mathrm{Tr}(\gamma\alpha^{\mathrm{p}^3}).
\end{aligned}$$

Clearly, (10) does not have then any solution in $\mathbb{F}_{p^4}$ if and only if $\mathrm{Tr}(\gamma\alpha^{p^3}) \neq -1-\mathrm{Tr}(\alpha\beta)$.

(ii) In the case $n = 6$, by Lemma 6, the equation $x + x^p = ka^{-p-1}\alpha - 1$ has solutions in $\mathbb{F}_{p^6}$ if and only if $\ell = a^{-p-1}\alpha + a^{-p^3-p^2}\alpha^{p^2} + a^{-p^5-p^4}\alpha^{p^4} = a^{-p^2-p}\alpha^p + a^{-p^4-p^3}\alpha^{p^3} + a^{-1-p^5}\alpha^{p^5} \in \mathbb{F}_p$. The right hand side of (10) equals then:

$$\begin{aligned}
\mathrm{RHS} &= \mathrm{Tr}\left(\gamma\alpha \mathrm{a}^{\mathrm{p}^3-\mathrm{p}} + \gamma\alpha^{\mathrm{p}^2}\mathrm{a}^{1-\mathrm{p}^2} - \gamma\alpha^{\mathrm{p}}\mathrm{a}^{\mathrm{p}^3+1-\mathrm{p}^2-\mathrm{p}}\right) \\
&= \mathrm{Tr}\left(\gamma\alpha \mathrm{a}^{\mathrm{p}^3-\mathrm{p}} + \gamma\alpha^{\mathrm{p}^2}\mathrm{a}^{1-\mathrm{p}^2} - \gamma\mathrm{a}^{\mathrm{p}^3+1}(\ell - \mathrm{a}^{-\mathrm{p}^4-\mathrm{p}^3}\alpha^{\mathrm{p}^3} - \mathrm{a}^{-1-\mathrm{p}^5}\alpha^{\mathrm{p}^5})\right) \\
&= \mathrm{Tr}\left(\gamma\alpha \mathrm{a}^{\mathrm{p}^3-\mathrm{p}} + \gamma\alpha^{\mathrm{p}^2}\mathrm{a}^{1-\mathrm{p}^2} - \ell\gamma\mathrm{a}^{\mathrm{p}^3+1} + \gamma\alpha^{\mathrm{p}^3}\mathrm{a}^{1-\mathrm{p}^4} + \gamma\alpha^{\mathrm{p}^5}\mathrm{a}^{\mathrm{p}^3-\mathrm{p}^5}\right) \\
&= \mathrm{Tr}\left((\gamma\alpha^{\mathrm{p}^3} + (\gamma\alpha)^{\mathrm{p}^3})\mathrm{a}^{1-\mathrm{p}^4} + (\gamma\alpha^{\mathrm{p}^2} + (\gamma\alpha^{\mathrm{p}^5})^{\mathrm{p}^3})\mathrm{a}^{1-\mathrm{p}^2} - \ell\gamma\mathrm{a}^{\mathrm{p}^3+1}\right) \\
&= \mathrm{Tr}\left((\gamma + \gamma^{\mathrm{p}^3})(\alpha^{\mathrm{p}^3}\mathrm{a}^{1-\mathrm{p}^4} + \alpha^{\mathrm{p}^2}\mathrm{a}^{1-\mathrm{p}^2}) - \ell\gamma\mathrm{a}^{\mathrm{p}^3+1}\right).
\end{aligned}$$

Since $\gamma^{p^3-1} = -1$ and hence $\gamma^{p^3} = -\gamma$, from the above equation we have RHS $= \mathrm{Tr}(-\ell\gamma\mathrm{a}^{\mathrm{p}^3+1}) = -\ell\mathrm{Tr}(\gamma\mathrm{a}^{\mathrm{p}^3+1}) = -\ell(\gamma\mathrm{a}^{\mathrm{p}^3+1} - \gamma\mathrm{a}^{\mathrm{p}^3+1} + \gamma\mathrm{a}^{\mathrm{p}^3+1} - \gamma\mathrm{a}^{\mathrm{p}^3+1} + \gamma\mathrm{a}^{\mathrm{p}^3+1} - \gamma\mathrm{a}^{\mathrm{p}^3+1}) = 0$. Hence, if $-1 - \mathrm{Tr}(\alpha\beta) \neq 0$, (10) does not have solutions and we complete the proof. □

Remark 4. *1. The functions of Theorem 1 are of the form $G(x^{p+1})$. They satisfy Corollary 1. Showing this would result in a similar but slightly more complex proof.*
2. The condition on γ in Theorem 1 (ii) is equivalent to: "$\gamma^2 \in \mathbb{F}^*_{p^3}$ and $\gamma \notin \mathbb{F}^*_{p^3}$".
3. For $p = 3$ and $n = 4$, by visiting exhaustively all α, β and γ in $\mathbb{F}_{p^n}$ satisfying the condition in Theorem 1 (i), we get only one class under CCZ equivalence, namely x^4. It is interesting to mention that Theorem 2 in Section 5 will show that the image of any quadratic zero-difference 3-balanced function on $\mathbb{F}_{3^4}$ is a $(81, 20, 1, 6)$-SRG. We know from [1] that there is only one such graph up to isomorphism.
4. For $p = 5, n = 4$, we could only perform partial search of coefficients $\alpha, \beta, \gamma \in \mathbb{F}_{5^4}$; we also get one class under CCZ inequivalence of quadratic zero-difference 5-balanced function, namely x^6.
5. For $n = 8, 10$, by a computer search, we did not find other coefficients α, β, γ such that F in Theorem 1 is a quadratic zero-difference p-balanced function, except for $\alpha = 0$.
6. In Theorem 1(ii), there exist coefficients α, β, γ which do not satisfy the conditions in the Theorem, while the function F is still a zero-difference p-balanced function.

5 Strongly regular graphs and quadratic zero-difference p^t-balanced functions

In this section, we discuss the relationship between partial differential sets and quadratic zero-difference p^t-balanced functions, which are of the form $F(x) = G(x^{p^t+1})$ where $G|_{C_{p^t+1}}$ is an injection. We recall first a well-known fact and we give a lemma which is used to prove Theorem 2 below.

Lemma 7 ([34]). *An algebraic integer $X \in \mathbb{Z}[\zeta_p]$ is a rational integer if and only if $\sigma_\alpha(X) = X$ for all $\alpha \in \mathbb{F}_p$ with $\alpha \neq 0$, where $\sigma_\alpha \in Gal(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ defined by $\sigma_\alpha(\zeta_p) = \zeta_p^\alpha$.*

In the following, for any nonzero integer a, we use the notation $\mathrm{ord}_2(a)$ to denote the highest integer t such that $2^t \mid a$ and $2^{t+1} \nmid a$.

Lemma 8. *Let p be a prime and t, n be two positive integers. Let $a = \mathrm{ord}_2(p^t+1), b = \mathrm{ord}_2(p^t-1), c = \mathrm{ord}_2(p^n-1)$ and $d = \mathrm{ord}_2(p^n+1)$. Then:*

(i) $\gcd(p^t+1, p^n-1) = \delta_{t,n} \cdot \frac{p^{\gcd(2t,n)}-1}{p^{\gcd(t,n)}-1}$, *where* $\delta_{t,n} = 2^{\min(a,c)+\min(b,c)-\min(a+b,c)} \in \{1,2\}$. *Furthermore, $p^t+1 \mid p^n-1$ if and only if $2t \mid n$;*

(ii) $\gcd(p^t+1, p^n+1) = \eta_{t,n} \cdot \frac{p^{\gcd(2t,2n)}-1}{p^{\gcd(t,2n)}-1} \cdot \frac{p^{\gcd(t,n)}-1}{p^{\gcd(2t,n)}-1}$, *where* $\eta_{t,n} = \frac{\eta'_{t,n}\delta_{t,2n}}{\delta_{t,n}} \in \{1,2\}$ *and* $\eta'_{t,n} = 2^{\min(a,c)+\min(a,d)-\min(a,c+d)}$. *Furthermore, $p^t+1 \mid p^n+1$ if and only if $n = \ell t$ for some odd integer ℓ.*

(iii) Let $n = 2kt$ for some positive integer k. Then, for every positive integer i, we have $p^t+1 \mid p^{n/2+i}-1$ if and only if $i \equiv kt \mod 2t$; and $p^t+1 \mid p^{n/2+i}+1$ if and only if $i \equiv (k+1)t \mod 2t$.

Proof. (i) First note that, since p^t-1 and p^t+1 have gcd equal to 1 if $p=2$ and to 2 if p is odd, we have $\gcd(p^t+1, p^n-1)\gcd(p^t-1, p^n-1) = \delta_{t,n}\gcd((p^t+1)(p^t-1), p^n-1))$, where $\delta_{t,n}$ equals 1 if $p=2$ and is a power of 2 if p is odd. It is a simple matter to see more precisely that $\delta_{t,n}$ equals the value defined above. Since $\gcd((p^t+1)(p^t-1), p^n-1)) = \gcd(p^{2t}-1, p^n-1) = p^{\gcd(2t,4n)}-1$, this proves the value of $\gcd(p^t+1, p^n-1)$. The fact that $\delta_{t,n} \in \{1,2\}$ is obvious when $p=2$. For p odd, from $\gcd(p^t+1, p^t-1) = 2$, we have $\min(a,b) = 1$, where a, b are defined above. If $a = 1$, then $\min(a,c) + \min(b,c) - \min(a+b,c) = 1 + \min(b,c) - \min(b+1,c)$ equals either 0 or 1, and therefore $\delta_{t,n} = 1$ or 2. Similarly we may show the case $b = 1$.
It is straightforward that, if $2t \mid n$ then the value obtained for $\gcd(p^t+1, p^n-1)$ equals p^t+1. Conversely:
(1) If $\mathrm{ord}_2(t) \geq \mathrm{ord}_2(n)$ then $\gcd(2t, n) = \gcd(t, n)$ and hence $\gcd(p^t+1, p^n-1) = \delta_{t,n}$. Since $\delta_{t,n} \in \{1,2\}$ and $p^t+1 > 2$, $p^t+1 \mid p^n-1$ cannot happen.
(2) If $\mathrm{ord}_2(t) < \mathrm{ord}_2(n)$ then $\gcd(p^t+1, p^n-1) = \delta_{t,n}(p^{\gcd(t,n)}+1)$. It is easy to check that in this case if $t \nmid n$, then $p^t+1 \nmid p^n-1$ as $\delta_{t,n}(p^{\gcd(t,n)}+1) \neq p^t+1$, since $(p^t+1)/(p^{\gcd(t,n)}+1) \neq 1, 2$. Therefore we have $t \mid n$. Further, one may check in this case we have $2t \mid n$.

(ii) Similarly to the beginning of (i), since p^n-1 and p^n+1 have gcd equal to 1 if $p=2$ and to 2 if p is odd, we have $\gcd(p^t+1, p^n+1)\gcd(p^t+1, p^n-1) = \eta'_{t,n}\gcd(p^t+$

$1, (p^n-1)(p^n+1))$, where $\eta'_{t,n} = 2^{\min(a,c)+\min(a,d)-\min(a,c+d)}$. Using (i), we get

$$\begin{aligned}\gcd(p^t+1, p^n+1) &= \eta'_{t,n}\frac{\gcd(p^t+1,p^{2n}-1)}{\gcd(p^t+1,p^n-1)} \\ &= \eta'_{t,n}\left(\delta_{t,2n}\cdot\frac{p^{\gcd(2t,2n)}-1}{p^{\gcd(t,2n)}-1}\right)\left(\frac{1}{\delta_{t,n}}\cdot\frac{p^{\gcd(t,n)}-1}{p^{\gcd(2t,n)-1}}\right) \qquad (11)\\ &= \frac{\eta'_{t,n}\delta_{t,2n}}{\delta_{t,n}}\cdot\frac{(p^{\gcd(2t,2n)}-1)(p^{\gcd(t,n)}-1)}{(p^{\gcd(t,2n)}-1)(p^{\gcd(2t,n)}-1)}.\end{aligned}$$

This completes the proof of the value of $\gcd(p^t+1, p^n+1)$. It is tedious but easy to show that $\eta_{t,n} \in \{1, 2\}$.
(1) If $\mathrm{ord}_2(t) = \mathrm{ord}_2(n)$, then from (11) we have

$$\gcd(p^t+1, p^n+1) = \eta_{t,n}\cdot\left(p^{\gcd(t,n)}+1\right).$$

It is easy to see that if $t \nmid n$ then $p^t+1 \nmid p^n+1$ since $(p^t+1)/(p^{\gcd(t,n)}+1) \neq 1, 2$. Hence we have $t \mid n$ and furthermore in this case n/t is odd.
(2) If $\mathrm{ord}_2(t) \neq \mathrm{ord}_2(n)$, then one may easily verify that in this case $\gcd(p^t+1, p^n+1) = \eta_{t,n}$. Since $\eta_{t,n} \in \{1, 2\}$ and $p^t+1 > 2$, in this case $p^t+1 \nmid p^n+1$.
Conversely, if $t \mid n$ and n/t is odd, one may easily verify that $p^t+1 \mid p^n+1$.

(iii) By (i), we have $p^t+1 \mid p^{n/2+i}-1$ if and only if $2t \mid n/2+i$, i.e. $i \equiv -kt \mod 2t \equiv kt \mod 2t$ we proved then the first part. Next, by (ii), we have $p^t+1 \mid p^{n/2+i}+1$ if and only if $n/2+i = kt+i = \ell t$ for some odd integer ℓ. Equivalently, $i \equiv (k+1)t \mod 2t$. This completes the proof. □

Lemma 9. *Let $F : \mathbb{F}_{p^n} \mapsto \mathbb{F}_{p^n}$ be defined as $F(x) = G(x^{p^t+1})$, for some function G over $\mathbb{F}_{p^n}$. If F is quadratic, then F can be represented in the form $F(x) = \sum_{i\in\Omega} a_i x^{p^{k_i}+p^{\ell_i}}$, where Ω is some subset of $\mathbb{N}$ and, for every $i \in \Omega$, $t \mid (k_i - \ell_i)$ and $(k_i-\ell_i)/t$ is odd.*

Proof. Since $F(x) = G(x^{p^t+1})$, then we have $F(x) = \sum_{j\geq 0} b_j x^{j(p^t+1)}$, where $b_j \in \mathbb{F}_{p^n}$ (note that we do not bound here the values of the exponents since we do not reduce modulo $x^{p^n}-x$). Since F is quadratic, we have $\sum_{j\geq 0} b_j x^{j(p^t+1)} = \sum_{i\in\Omega} a_i x^{p^{k_i}+p^{\ell_i}}$, where $a_i \in \mathbb{F}_{p^n}$ and Ω is a subset of $\mathbb{N}$ (note indeed that the form $p^{k_i}+p^{\ell_i}$ of the exponents is invariant under congruence modulo p^n-1). We can assume that $p^{k_i} \geq p^{\ell_i}$ for each i in Ω. For each i in Ω, we have $p^t+1 \mid p^{k_i}+p^{\ell_i} = p^{\ell_i}(p^{k_i-\ell_i}+1)$. Hence $p^t+1 \mid p^{k_i-\ell_i}+1$. By Lemma 8(ii), we obtain $t \mid k_i-\ell_i$ and $(k_i-\ell_i)/t$ is odd. The proof is completed. □

Lemma 10. *Let $F(x) = G(x^d)$ be a quadratic function from $\mathbb{F}_{p^n}$ to itself, where p is any prime, $n = 2kt$ and $\gcd(d, p^n-1) = p^t+1$ for some non-negative integer t. For each nonzero $a \in \mathbb{F}_{p^n}$, define*

$$E_a = \{s : s \in \mathbb{F}_{p^n} | \mathrm{Tr(a(F(y+s)-F(y)))} = 0 \text{ for all } y \in \mathbb{F}_{p^n}\}.$$

Then E_a is a vector space over $\mathbb{F}_p$ with even dimension.

Proof. Clearly $E_a \neq \emptyset$ as $0 \in E_a$. Moreover, E_a is a vector space over $\mathbb{F}_p$ since if $\mathrm{Tr}(\mathrm{a}(\mathrm{F}(\mathrm{y}+\mathrm{s})-\mathrm{F}(\mathrm{y})))$ and $\mathrm{Tr}(\mathrm{a}(\mathrm{F}(\mathrm{y}+\mathrm{s}')-\mathrm{F}(\mathrm{y})))$ both equal the null function, then for every $u, v \in \mathbb{F}_p$, $\mathrm{Tr}(\mathrm{a}(\mathrm{F}(\mathrm{y}+\mathrm{us})-\mathrm{F}(\mathrm{y})))$ and $\mathrm{Tr}(\mathrm{a}(\mathrm{F}(\mathrm{y}-\mathrm{vs}')-\mathrm{F}(\mathrm{y})))$ are also null and by substraction $\mathrm{Tr}(\mathrm{a}(\mathrm{F}(\mathrm{y}+\mathrm{us})-\mathrm{F}(\mathrm{y}-\mathrm{vs}')))$ is null and then $\mathrm{Tr}(\mathrm{a}(\mathrm{F}(\mathrm{y}+\mathrm{us}+\mathrm{vs}')-\mathrm{F}(\mathrm{y})))$ is the null function.

When $p = 2$, it is well-known (see e.g. [7]) that the dimension of E_a is even. When p is an odd prime, let us show that E_a is a vector space over $\mathbb{F}_{p^2}$, and therefore, the dimension of E_a as a vector space over $\mathbb{F}_p$ is even.

By Lemma 9, F has the form $F(x) = \sum_{i \in \Omega} a_i x^{p^{k_i}+p^{\ell_i}}$, where $t \mid k_i - \ell_i$ and $(k_i - \ell_i)/t$ is odd for each $i \in \Omega$. An element s of $\mathbb{F}_{p^n}$ belongs to E_a if and only if:

$$\forall y \in \mathbb{F}_{p^n},\ \mathrm{Tr}(\mathrm{a}(\mathrm{F}(\mathrm{y}+\mathrm{s})-\mathrm{F}(\mathrm{y}))) = 0. \tag{12}$$

Substituting $F(x) = \sum_{i \in \Omega} a_i x^{p^{k_i}+p^{\ell_i}}$ into (12) and simplifying, we have then:

$$\forall y \in \mathbb{F}_{p^n},\ \mathrm{Tr}\left(\mathrm{a}\sum_{\mathrm{i}} \mathrm{a_i}(\mathrm{s}^{\mathrm{p}^{\mathrm{k_i}}+\mathrm{p}^{\ell_\mathrm{i}}} + \mathrm{s}^{\mathrm{p}^{\mathrm{k_i}}}\mathrm{y}^{\mathrm{p}^{\ell_\mathrm{i}}} + \mathrm{s}^{\mathrm{p}^{\ell_\mathrm{i}}}\mathrm{y}^{\mathrm{p}^{\mathrm{k_i}}})\right) = 0.$$

Therefore we have $s \in E_a$ if and only if, for every $y \in \mathbb{F}_{p^n}$:

$$\begin{aligned} -\mathrm{Tr}\left(\sum_{\mathrm{i}} \mathrm{aa_i}\mathrm{s}^{\mathrm{p}^{\mathrm{k_i}}+\mathrm{p}^{\ell_\mathrm{i}}}\right) &= \mathrm{Tr}\left(\sum_{\mathrm{i}} \mathrm{aa_i}(\mathrm{s}^{\mathrm{p}^{\mathrm{k_i}}}\mathrm{y}^{\mathrm{p}^{\ell_\mathrm{i}}} + \mathrm{s}^{\mathrm{p}^{\ell_\mathrm{i}}}\mathrm{y}^{\mathrm{p}^{\mathrm{k_i}}})\right) \\ &= \mathrm{Tr}\left(\sum_{\mathrm{i}} \left((\mathrm{aa_i}\mathrm{s}^{\mathrm{p}^{\mathrm{k_i}}})^{\mathrm{p}^{-\ell_\mathrm{i}}} + (\mathrm{aa_i}\mathrm{s}^{\mathrm{p}^{\ell_\mathrm{i}}})^{\mathrm{p}^{-\mathrm{k_i}}}\right)\mathrm{y}\right). \end{aligned} \tag{13}$$

Note that the left hand side of (13) equals $-\mathrm{Tr}(\mathrm{aF}(\mathrm{s}))$ and is null by letting $y = 0$ in (12) (we may assume without loss of generality that $F(0) = 0$ as otherwise we may replace $F(x)$ with $F'(x) = F(x) - F(a)$ and F' satisfies all properties of F which are stated in the hypothesis). The condition becomes:

$$\sum_i \left((aa_i s^{p^{k_i}})^{p^{-\ell_i}} + (aa_i s^{p^{\ell_i}})^{p^{-k_i}}\right) = 0. \tag{14}$$

Recall that we have $t \mid (k_i - \ell_i)$ and $(k_i - \ell_i)/t$ is odd. In the following let $k_i - \ell_i = e_i t$ for i in Ω, where e_i is some odd integer. Then (14) becomes:

$$\sum_{i=0}^{n-1} \left((aa_i)^{p^{-\ell_i}} s^{p^{e_i t}} + (aa_i)^{p^{-k_i}} s^{p^{-e_i t}}\right) = 0. \tag{15}$$

For any $s \in E_a$ and any $w \in \mathbb{F}_{p^2}$, we have $ws \in E_a$. Indeed, w^{p^e} equals w if e is even and equals w^p if e is odd. Then $(aa_i)^{p^{-\ell_i}}(ws)^{p^{e_i t}} + (aa_i)^{p^{-k_i}}(ws)^{p^{-e_i t}} = w^{p^t}[(aa_i)^{p^{-\ell_i}} s^{p^{e_i t}} + (aa_i)^{p^{-k_i}} s^{p^{-e_i t}}] = 0$. For any constants $w_1, w_2 \in \mathbb{F}_{p^2}$ and any $s_1, s_2 \in E_a$, we have then $w_1 s_1 + w_2 s_2 \in E_a$, since $\mathrm{Tr}(\mathrm{a}(\mathrm{F}(\mathrm{y}+\mathrm{w_1 s_1})-\mathrm{F}(\mathrm{y})))$ and $\mathrm{Tr}(\mathrm{a}(\mathrm{F}(\mathrm{y}-\mathrm{w_2 s_2})-\mathrm{F}(\mathrm{y})))$ are constant zero and by substraction $\mathrm{Tr}(\mathrm{a}(\mathrm{F}(\mathrm{y}+\mathrm{w_1 s_1})-\mathrm{F}(\mathrm{y}-\mathrm{w_2 s_2})))$ is constant zero and then $\mathrm{Tr}(\mathrm{a}(\mathrm{F}(\mathrm{y}+\mathrm{w_1 s_1}+\mathrm{w_2 s_2})-\mathrm{F}(\mathrm{y})))$ is constant zero. □

We will need the following result in [22, Lemma 3] for the proof of the main theorem.

Lemma 11. *Let f be a quadratic perfect nonlinear function with $f(-x) = f(x)$ for all nonzero $x \in \mathbb{F}_{p^n}$ and $f(0) = 0$. Then the Walsh coefficient $\mathcal{W}_{\mathrm{Tr(aF)}}(0) = \epsilon_{a,0}(\sqrt{p^*})^n$, where $\epsilon_{a,0} \in \{\pm 1\}$, $p^* = \left(\frac{-1}{p}\right) p$ and $\left(\frac{-1}{p}\right)$ is the Legendre symbol.*

Now we are ready to give the main result of this Section. Note that the first part of the following result first appeared in [42]. We include it here and give a short proof for the completeness of the paper. Note that the second part of the following result may give rise to new SRGs by using certain ZDB functions and comparing them with the known constructions in [2, 3, 32].

Theorem 2. *Let $F(x) = G(x^d)$ be a quadratic function from $\mathbb{F}_{p^n}$ to itself, where p is any prime and $\gcd(d, p^n - 1) = p^t + 1$ for some non-negative integer t. Assume that the restriction of G to $C_d = \{x^d : x \in \mathbb{F}_{p^n}^*\} = C_{p^t+1}$ is an injection from C_d to $\mathbb{F}_{p^n}$. Define the set $D = \{F(x) : x \in \mathbb{F}_{p^n}\} \setminus \{0\}$. Then:*

(i) if $t = 0$ and p is an odd prime, then D is a

$$\begin{aligned} \left(p^n, \tfrac{p^n-1}{2}, \tfrac{p^n-3}{4}\right) &\text{ difference set,} && \text{when } p^n \equiv 3 \mod 4, \\ \left(p^n, \tfrac{p^n-1}{2}, \tfrac{p^n-5}{4}, \tfrac{p^n-1}{4}\right) &\text{ partial difference set,} && \text{when } p^n \equiv 1 \mod 4. \end{aligned}$$

(ii) if $t > 0$ and n is divisible by $2t$, then D is a

$$\left(p^n, \frac{p^n-1}{p^t+1}, \frac{p^n - 3p^t - 2 - \epsilon p^{n/2+2t} + \epsilon p^{n/2+t}}{(p^t+1)^2}, \frac{p^n - \epsilon p^{n/2} + \epsilon p^{n/2+t} - p^t}{(p^t+1)^2}\right)$$

partial difference set, where $n = 2kt$ and $\epsilon = (-1)^k$.

Proof. Without loss of generality, we may assume that $d = p^t + 1$. Let us denote the additive group of $\mathbb{F}_{p^n}$ by $\mathcal{G}$. By Corollary 1, to prove that D is a (partial) difference set with the prescribed parameters, we need to determine the character values of D. Now, for each nontrivial character $\chi_a \in \widehat{\mathcal{G}}, a \in \mathcal{G}^*$, we have $\chi_a(D) = \sum_{x \in C_d} \zeta_p^{\mathrm{Tr(aG(x))}}$, where ζ_p is the chosen p-th root of unity. It is not difficult to see that

$$\mathcal{W}_{\mathrm{Tr(aF)}}(0) = \sum_{x \in \mathbb{F}_{p^n}} \zeta_p^{\mathrm{Tr(aF(x))}} = 1 + d \sum_{x \in C_d} \zeta_p^{\mathrm{Tr(aG(x))}} = 1 + d\chi_a(D),$$

and hence

$$\chi_a(D) = \frac{1}{d}\left(\mathcal{W}_{\mathrm{Tr(aF)}}(0) - 1\right). \tag{16}$$

Denoting $\mathcal{W}_{\mathrm{Tr(aF)}}(0)$ by X_a, we have

$$X_a \overline{X_a} = \sum_{x,y \in \mathbb{F}_{p^n}} \zeta_p^{\mathrm{Tr(a(F(x)-F(y)))}} = \sum_{t \in \mathbb{F}_{p^n}} \left(\sum_{y \in \mathbb{F}_{p^n}} \zeta_p^{\mathrm{Tr(a(F(y+t)-F(y)))}} \right). \tag{17}$$

(i): For $t=0$ we have $d=p^0+1=2$. By hypothesis, we assume that $F(x)=G(x^2)$ is a quadratic function and that $G|_{C_2}$ is an injection. Then F is a PN function (see [42] or Corollary 1 above) and then $F(y+t)-F(y)$ is a PP over $\mathbb{F}_{p^n}$ for any nonzero t. Therefore, by (17) we have $X_a\overline{X_a}=p^n$. By Lemma 11, we have $X_a=\varsigma(\sqrt{p^*})^n$, where $\varsigma\in\{-1,1\}$ and $p^*=\left(\frac{-1}{p}\right)p$. In the following we divide the proof into two cases.

Case 1: n is even. Note that in this case $p^n\equiv 1 \mod 4$, which implies that $X_a=\varsigma(\sqrt{p^*})^n=\varsigma\left((\frac{-1}{p})p\right)^{n/2}=\varsigma(\frac{-1}{p})^{n/2}p^{n/2}$. Hence we have $\chi_a(D)=\frac{1}{2}(\varsigma(\frac{-1}{p})^{n/2}p^{n/2}-1)$. It can be verified that $\chi_a(DD^{(-1)})=\chi_a(D)\chi_a(D^{(-1)})=\chi_a(D)\overline{\chi_a(D)}=\frac{1}{4}(p^n+1-2\varsigma(\frac{-1}{p})^{n/2}p^{n/2})$. On the other hand, it can be easily computed that $(k-\lambda)+(\mu-\lambda)\chi_a(D)=\frac{1}{4}(p^n+1-2\varsigma(\frac{-1}{p})^{n/2}p^{n/2}))$. Then, by Lemma 3, we have that D is a PDS with the prescribed parameter.

Case 2: $p\equiv 3 \mod 4$ and n is odd. Assume that $n=2m+1$. In this case we have $X_a=\varsigma(\sqrt{p^*})^{2m+1}=\varsigma\left((\frac{-1}{p})p\right)^m\sqrt{p^*}=\varsigma\left(\frac{-1}{p}\right)^m p^m\sqrt{p^*}$, then $\chi_a(D)=\frac{1}{2}(\varsigma\left(\frac{-1}{p}\right)^m p^m\sqrt{p^*}-1)$. On the one hand, note that the complex conjugate of $\sqrt{p^*}$ equals $-\sqrt{p^*}$ (since $\sqrt{p^*}\cdot(-\sqrt{p^*})=-p^*=-\left(\frac{-1}{p}\right)p=p=|\sqrt{p^*}|^2$). Then $\chi(DD^{(-1)})=\chi(D)\overline{\chi(D)}=\frac{1}{4}(\varsigma\left(\frac{-1}{p}\right)^m p^m\sqrt{p^*}-1)(\varsigma\left(\frac{-1}{p}\right)^m p^m\overline{\sqrt{p^*}}-1)=\frac{1}{4}(\varsigma\left(\frac{-1}{p}\right)^m p^m\sqrt{p^*}-1)(-\varsigma\left(\frac{-1}{p}\right)^m p^m\sqrt{p^*}-1)=-\frac{1}{4}(\left(\frac{-1}{p}\right)p^n-1)=\frac{1}{4}(p^n+1)$ (since $\left(\frac{-1}{p}\right)=-1$ as $p\equiv 3 \mod 4$). On the other hand, one may compute that $k-\lambda=\frac{1}{4}(p^n+1)$. By Lemma 3, we prove that D is the difference set with the prescribed parameters.

(ii) Next, we deal with the case that $d=p^t+1$ with $t>0$. First we claim that $D=D^{(-1)}$ in this case. When $p=2$, this is clear. When p is an odd prime, we have $p^n\equiv 1 \mod 4$ since n is even and then -1 is a square in $\mathbb{F}_{p^n}$. Since $2(p^t+1)\mid p^n-1$ (one may see that $p^n-1=p^{2kt}-1=(p^t+1)(1-p^t+p^{2t}+\cdots+(-1)^{(2k-1)}p^{(2k-1)t})$ and the second factor is even), there exists an element $\alpha\in\mathbb{F}_{p^n}$ such that its order is $2(p^t+1)$, namely $\alpha^{p^t+1}=-1$. By Lemma 9, we have $F(\alpha x)=\sum_{i\in\Omega}a_i(\alpha x)^{p^{k_i}+p^{\ell_i}}=\sum_{i\in\Omega}a_i(\alpha x)^{p^{\ell_i}(p^{e_it}+1)}$, where $e_i=(k_i-\ell_i)/t$ is odd by Lemma 9. Since $\alpha^{p^{\ell_i}(p^{e_it}+1)}=\alpha^{p^{\ell_i}(p^t+1)(1-p^t+\cdots+(-1)^{e_i-1}p^{(e_i-1)t})}=\left(\alpha^{p^t+1}\right)^{p^{\ell_i}(1-p^t+\cdots+(-1)^{e_i-1}p^{(e_i-1)t})}=-1$. Therefore from the above computations we get $F(\alpha x)=-F(x)$ for every $x\in\mathbb{F}_{p^n}$, which implies that $D=D^{(-1)}$.

Further, since F is quadratic, the function $f_{a,s}(y)=\mathrm{Tr(a(F(y+s)-F(y)))}$ is affine for every $a,s\in\mathbb{F}_{p^n}$ and the sum $\sum_{y\in\mathbb{F}_{p^n}}\zeta_p^{\mathrm{Tr(a(F(y+s)-F(y)))}}$ is then nonzero only when $f_{a,s}(y)$ is a constant function. For each $a\in\mathcal{G}^*$, let us define then

$$E_a=\{s:s\in\mathbb{F}_{p^n}|\mathrm{Tr(a(F(y+s)-F(y)))\ is\ a\ constant\ for\ all\ y}\in\mathbb{F}_{\mathrm{p^n}}\}.$$

According to Corollary 1, for each nonzero s, there always exists $y\in\mathbb{F}_{p^n}$ such that $F(y+s)-F(y)=0$. As a result, the set E_a actually is

$$E_a=\{s:s\in\mathbb{F}_{p^n}|\mathrm{Tr(a(F(y+s)-F(y)))=0\ for\ all\ y}\in\mathbb{F}_{\mathrm{p^n}}\}$$

and Relation (17) becomes $X_a\overline{X_a} = p^n|E_a|$.
By Lemma 10, we know that E_a is a vector space over $\mathbb{F}_p$ and its dimension is even. For all $0 \le i \le n/2$, let N_i be the number of nonzero $a \in \mathbb{F}_{p^n}$ such that $|E_a| = p^{2i}$. Now we consider the following sum,

$$\sum_{a\in\mathbb{F}_{p^n}^*} X_a\overline{X_a} = \sum_{a\in\mathbb{F}_{p^n}^*}\sum_{x,y\in\mathbb{F}_{p^n}} \zeta_p^{\mathrm{Tr(a(F(x)-F(y)))}}. \tag{18}$$

On the one hand, the LHS of (18) equals $\sum_{a\in\mathbb{F}_{p^n}^*} p^n|E_a| = \sum_{i\ge 0} p^{n+2i}N_i$. On the other hand, the RHS of (18) is

$$\begin{aligned}\text{RHS} &= \sum_{x,y\in\mathbb{F}_{p^n}}\sum_{a\in\mathbb{F}_{p^n}} \zeta_p^{\mathrm{Tr(a(F(x)-F(y)))}} - p^{2n} = p^n|\{(x,y)\in\mathbb{F}_{p^n}^2 | F(x)=F(y)\}| - p^{2n}\\ &= p^n(1+d(p^n-1)) - p^{2n} = (d-1)p^n(p^n-1) = (d-1)p^n\sum_{i\ge 0} N_i.\end{aligned}$$

Hence, we have $\sum_{i\ge0} p^{n+2i}N_i = (d-1)p^n\sum_{i\ge0} N_i$, which implies that $\sum_{i\ge0}(p^{2i}-(d-1))N_i = \sum_{i\ge0}(p^{2i}-p^t)N_i = 0$. Rearrange the term, we get

$$(p^t-1)N_0 = \sum_{i\ge1}(p^{2i}-p^t)N_i. \tag{19}$$

Since $D = D^{(-1)}$ and $\chi_a(D) \in \mathbb{Z}[\zeta_p]$, we have $\chi_a(D) \in \mathbb{Z}$ by Lemma 7 and then $X_a = d\chi_a(D)+1 \in \mathbb{Z}$. Therefore, from $X_a\overline{X_a} = p^{n+2i}$ for some $i \ge 0$ we get $X_a = \pm p^{n/2+i}$ since n is even. Furthermore, by Lemma 8(iii) and $\chi_a(D) = \frac{1}{p^t+1}(X_a-1) = \frac{1}{p^t+1}(\pm p^{n/2+i}-1) \in \mathbb{Z}$, we have that $N_i \ne 0$ if and only if $i \mod 2t \in \{kt,(k+1)t\}$, and

$$(X_a,\chi_a(D)) = \begin{cases} \left(p^{n/2+i}, \frac{p^{n/2+i}-1}{p^t+1}\right) & \text{if } i\equiv kt \mod 2t,\\ \left(-p^{n/2+i}, \frac{-p^{n/2+i}-1}{p^t+1}\right) & \text{if } i\equiv (k+1)t \mod 2t.\end{cases} \tag{20}$$

Let Φ be the set $\{0\le i\le n/2 | i \mod 2t \in \{kt,(k+1)t\}\}$. Hence, since $\sum_{a\in\mathbb{F}_{p^n}^*} X_a = \sum_{a\in\mathbb{F}_{p^n}^*}\sum_{x\in\mathbb{F}_{p^n}} \zeta_p^{\mathrm{Tr(aF(x))}} = p^n|\{x\in\mathbb{F}_{p^n} \mid F(x)=0\}| - p^n = 0$ we have $\sum_{i\in\Phi}(-p)^{(n/2+i)/t}N_i = 0$, that is:

$$\sum_{\substack{i\in\Phi\\ i\equiv kt \mod 2t}} p^iN_i = \sum_{\substack{i\in\Phi\\ i\equiv (k+1)t \mod 2t}} p^iN_i. \tag{21}$$

By equation (21) we have $\sum_{i\in\Phi, i\equiv 0 \mod 2t} p^iN_i = \sum_{i\in\Phi, i\equiv t \mod 2t} p^iN_i$ and then $N_0 = \sum_{i\in\Phi,i\ne0}(-1)^{i/t-1}p^iN_i$. By equation (19) we have $N_0 = \sum_{i\in\Phi,i\ne0}\frac{p^{2i}-p^t}{p^t-1}N_i$. Note that one may easily check that $\frac{p^{2i}-p^t}{p^t-1} \ge p^i$ when $i \ge t$ and the equality holds if and only if $i = t$. Now, if N_i is nonzero for any $i \in \Phi\setminus\{0,t\}$, we will have

$$N_0 = \sum_{i\in\Phi,i\ne0}\frac{p^{2i}-p^t}{p^t-1}N_i > \sum_{i\in\Phi,i\ne0}(-1)^{i/t-1}p^iN_i = N_0,$$

which is a contradiction. Therefore we have $|E_a| = 1$ or p^{2t} for any $a \in \mathbb{F}_{p^n}^*$, and then the values of X_a lie in the set $\{\pm p^{n/2}, \pm p^{n/2+t}\}$. We divide the following proof into two cases according to the parity of k.

Case 1: k is even. By (20), we have $X_a \in \{p^{n/2}, -p^{n/2+t}\}$ and $\chi_a(D) \in \{\frac{p^{n/2}-1}{p^t+1}, \frac{-p^{n/2+t}-1}{p^t+1}\}$.

Case 2: k is odd. Again, by (20), we have $X_a \in \{-p^{n/2}, p^{n/2+t}$ and $\chi_a(D) \in \{-\frac{p^{n/2}-1}{p^t+1}, \frac{p^{n/2+t}-1}{p^t+1}\}$.

Finally, by Lemma 3(ii) and the fact that $D = D^{(-1)}$, we have that D is a PDS with the prescribed parameters. We complete the proof. □

Particularly, when $p = 2$ and $t = 1$, we may obtain PDS from APN functions.

Corollary 2. *Let F be a quadratic APN function on $\mathbb{F}_{2^n}$ with the form $F(x) = G(x^3)$, where $G|_{C_3}$ is an injection and $n = 2k$. Let D denote the set*

$$D = \{F(x) : x \in \mathbb{F}_{2^n}\} \setminus \{0\}.$$

Then D is a partial difference set with parameters

$$\big(2^n, \tfrac{2^n-1}{3}, \tfrac{1}{9}(2^k+4)(2^k-2), \tfrac{1}{9}(2^k+1)(2^k-2)\big) \quad \text{if } k \text{ is odd,}$$

$$\big(2^n, \tfrac{2^n-1}{3}, \tfrac{1}{9}(2^k-4)(2^k+2), \tfrac{1}{9}(2^k-1)(2^k+2)\big) \quad \text{if } k \text{ is even.}$$

From the proof of Theorem 2, we have the following result, which may be used to determine the Walsh spectrum of F in Theorem 2.

Corollary 3. *Let $F(x) = G(x^d)$ be a quadratic function from $\mathbb{F}_{p^n}$ to itself, where p is a prime, $\gcd(d, p^n - 1) = p^t + 1$ for some integer $t > 0$ and $n = 2kt$ for some positive integer k. Assume that the restriction of F to C_d is an injection from C_d to $\mathbb{F}_{p^n}$. For any $a \in \mathbb{F}_{p^n}^*$, define the set*

$$E_a = \{s : s \in \mathbb{F}_{p^n} | \mathrm{Tr}\,(\mathrm{a}(\mathrm{F}(\mathrm{y}+\mathrm{s}) - \mathrm{F}(\mathrm{y}))) = 0 \text{ for all } \mathrm{y} \in \mathbb{F}_{\mathrm{p^n}}\}.$$

Then $|E_a|$ is either 1 or p^{2t}.

Theorem 3. *Let $F(x) = G(x^d)$ be a quadratic function from $\mathbb{F}_{p^n}$ to itself, where p is a prime, $\gcd(d, p^n - 1) = p^t + 1$ for some integer $t > 0$ and $n = 2kt$ for some positive integer k. Then, for any $a, b \in \mathbb{F}_{p^n}$, the Walsh coefficient $\mathcal{W}_F(a, b)$ satisfies*

$$|\mathcal{W}_F(a, b)|^2 \in \{0, p^n, p^{n+2t}\}.$$

Particularly, if n is even and $F = G(x^3)$ is a quadratic APN function on $\mathbb{F}_{2^n}$, then the Walsh spectrum of F is $\{0, \pm 2^{n/2}, \pm 2^{n/2+1}\}$.

Proof. For any $a, b \in \mathbb{F}_{2^n}$, we have

$$\begin{aligned} \mathcal{W}_F(a,b)\overline{\mathcal{W}_F(a,b)} &= \sum_{x,y\in\mathbb{F}_{p^n}} \zeta_p^{\mathrm{Tr(a(F(x)-F(y))+b(x-y))}} \\ &= \sum_{t,x} \zeta_p^{\mathrm{Tr(aF(x+t)-aF(x)+bt)}} \\ &= \sum_{t} \zeta_p^{\mathrm{Tr(bt)}} \sum_{x} \zeta_p^{\mathrm{Tr(a(F(x+t)-F(x)))}} \\ &= p^n \sum_{t\in E_a} \zeta_p^{\mathrm{Tr(bt)}} = \chi_b(E_a), \end{aligned} \tag{22}$$

where χ_b is the character of $\mathbb{F}_{p^n}$. It is then clear that $\mathcal{W}_F(a,b)\overline{\mathcal{W}_F(a,b)}$ lies in the set $\{0, p^n|E_a|\}$ since E_a is an affine subspace. By Corollary 3, we have $\mathcal{W}_F(a,b)\overline{\mathcal{W}_F(a,b)} \in \{0, p^n, p^{n+2t}\}$. Particularly, when $F = G(x^3)$ is a quadratic APN function on $\mathbb{F}_{2^n}$ with n even, we have $\mathcal{W}_F(a,b) \in \mathbb{Z}$ and hence $\mathcal{W}_F(a,b) \in \{0, \pm p^{n/2}, \pm p^{n/2+1}\}$. The proof is completed. □

Remark 5. *The above result shows that the Walsh spectrum of the APN function with the form $F(x) = G(x^3)$ is the same as the one of Gold APN function. Theorem 3 presents a unifying treatment of determining the Walsh spectrum of quadratic APN functions with the form $F(x) = G(x^3)$, where G is an injection from C_3 to $\mathbb{F}_{2^n}$. For instance, it includes the result in [4] when n is even.*

6 Newness

In this section, we discuss the newness of the SRGs generated by the quadratic zero-difference p^t-balanced functions in Theorem 2. We first give some general results, and then we show that some of the obtained negative Latin square type SRGs are new by comparing them to the known constructions.

6.1 Isomorphism of graphs and equivalence of functions

We first introduce some definitions. Two graphs $\mathtt{G}_1 = (V_1, E_1)$ and $\mathtt{G}_2 = (V_2, E_2)$ are called *isomorphic* if there exists a one-to-one function σ mapping V_1 to V_2, and E_1 to E_2 such that for each pair $(P, e) \in V_1 \times E_1$, we have $\sigma(P) \in \sigma(e)$ if and only if $P \in e$. Let D_1, D_2 be two partial difference sets of the group $\mathcal{G}$, they are called *CI*-equivalent if there exists an automorphism $\phi \in \mathrm{Aut}(\mathcal{G})$ such that $\phi(D_1) = D_2$. Moreover, they are said to be *SRG*-equivalent if the corresponding Cayley graphs are isomorphic, i.e. $\mathrm{Cay}(\mathcal{G}, D_1) \cong \mathrm{Cay}(\mathcal{G}, D_2)$. It is known that SRG-equivalence implies CI-equivalence, but not vice versa (see an example in [25]). For more details on CI-equivalence and SRG-equivalence, one may refer to [42].

It is not difficult to see that the property of zero-difference balancedness for a function F is not necessarily preserved under EA-equivalence. However, the zero-difference balancedness property is preserved by affine equivalence and induces isomorphism of the associated Cayley graphs, but not the addition of a linear function. It is worth to point

out that EA-inequivalent functions may not necessarily lead to non-isomorphic graphs (see the following Table 1). Indeed, 18 EA-inequivalent APN functions with the form $G(x^3)$ on $\mathbb{F}_{2^8}$ were found in [41] (independently found in [39]), where G is an injection on the set of nonzero cubes (listed in the Appendix), the No. 2 and 6 functions, and the No. 13, 14 and 17 functions lead to isomorphic SRGs (see Table 1).

6.2 New negative Latin square type SRGs

It is well known that there are many non-isomorphic Latin square type SRGs by the following construction, therefore we only focus on the newness of negative Latin square type SRGs. Recall that the definition of (negative) Latin square type SRGs can be found at the end of Section 2.3.

Lemma 12 (PCP construction, [32]). *Let $\mathcal{G}$ be the additive group of a 2-dimensional vector space V over $\mathbb{F}_q$. Let $H_1, H_2, \ldots, H_r$, where $r \leq q+1$, be r hyperplanes of V. Then $D = (H_1 \cup H_2 \cup \cdots \cup H_r) \setminus \{0\}$ is a $(q^2, r(q-1), q+r^2-3r, r^2-r)$ partial difference set in $\mathcal{G}$.*

In the following we list the known constructions which may generate negative Latin square type SRGs with the same parameters in Theorem 2(ii). Recall that the e-th cyclotomic classes of a finite field $\mathbb{F}_{p^n}$ $(p^n = ef+1)$ are the subsets

$$C_j^e = \{w^{ie+j} : i = 0, \cdots, f-1\}, \quad (0 \leq j \leq e-1),$$

where w is a primitive element of $\mathbb{F}_{p^n}$.

Lemma 13 (Calderbank and Kantor, [6]). *Let q be a prime power and $C_0, C_1, \ldots, C_q$ be the $(q+1)$-th cyclotomic classes in $\mathbb{F}_{q^{2m}}$. For any $I \subset \{0, 1, \ldots, q\}$, $D = \cup_{i \in I} C_i$ is a regular $(q^{2m}, u(q^{2m}-1)/(q+1), u^2\eta^2 + (3u-q-1)\eta - 1, u^2\eta^2 + u\eta)$-PDS in the additive group of $\mathbb{F}_{q^{2m}}$ where $u = |I|$ and $\eta = ((-q)^m - 1)/(q+1)$.*

In [3, Theorem 2], Brouwer, Wilson and Xiang generalized the construction of SRGs in the above Lemma 13. Their construction requires the so-called semiprimitive condition. We shall elaborate below (before Table 1) that our constructions of SRGs in Theorem 2(ii) is more general since it does not require the semiprimitive condition and it may generate new SRGs which are not covered by [3, Theorem 2].

Finally, it is well known that negative Latin square type SRGs may be obtained from projective two-weight codes (see definition in [6]) as follows:

Lemma 14 ([32]). *Let $y_1, y_2, \ldots, y_n$ be pairwise linear independent vectors in $\mathbb{F}_q^n$. Then $y_1, y_2, \ldots, y_n$ span a two-weight $[n, s]$-projective code $\mathcal{C}$ if and only if*

$$D = \{ty_i : t \in \mathbb{F}_q \setminus \{0\} \text{ and } i = 1, 2, \ldots, n\}$$

is a regular PDS in the additive group of $\mathbb{F}_q^s$. Furthermore, if the two nonzero weights of $\mathcal{C}$ are w_1 and w_2, then D is a

$$(q^s, n(q-1), k^2 + 3k - q(k+1)(w_1+w_2) + q^2 w_1 w_2, k^2 + k - qk(w_1+w_2) + q^2 w_1 w_2)$$

partial difference set.

In the following, with the help of a computer, we show that, using the construction of SRGs in Theorem 2(ii) and the aforementioned 18 APN functions on $\mathbb{F}_{2^8}$, new negative Latin square type SRGs are obtained.

In Theorem 2, by letting $p = 2, t = 1, n = 8$, the 18 APN functions of the form $G(x^3)$ where $G|_{C_3}$ is an injection lead to $(256, 85, 24, 30)$-SRGs. By the online database of known constructions of SRGs [2], SRGs with these parameters can be constructed from the following methods:

(i) The SRG from Lemma 13 by letting $u = 1$. This graph is verified to be isomorphic to the SRGs with No. 13, 14, 17 APN functions in Appendix.

(ii) The SRG from projective binary $[85, 8]$ two-weight codes with weights $40, 48$ as in Lemma 14. Checking the online database of two-weight codes [12] shows that there are three constructions of such codes [13]. By a computer the SRGs from these codes are all isomorphic to the one from Lemma 13 above (actually by Magma these three codes are equivalent).

(iii) In [3, Theorem 2], to obtain an SRG with the above parameter, we need to require (using the same notation as in [3, Theorem 2]) $u/e = 1/3$, where $e \mid 255$ and there exists $l > 0$ such that $2^l \equiv -1 \mod e$ and $1 \leq u \leq e - 1$. It is easy to verify that for all divisors of 255 only $e = 3, u = 1$ satisfy the above conditions. Then by [3, Theorem 2], the Cayley graphs generated by the sets $D_i = \alpha^i K$ are SRGs with the parameters $(256, 85, 24, 30)$, where K is the set of all non-zero cubes of $\mathbb{F}_{2^8}$, α is a primitive element of $\mathbb{F}_{2^8}$ and $i \in \{0, 1, 2\}$. Clearly, K is exactly the image set of the APN function x^3 over $\mathbb{F}_{2^8}$. Therefore, the SRGs with the parameters $(256, 85, 24, 30)$ from [3, Theorem 2] are isomorphic to the one from our Theorem 2(ii) by applying $F(x) = x^3$.

By MAGMA, the SRGs from the other 15 APN functions in the Appendix are not isomorphic to the known constructions, and pairwise non-isomorphic. We list the 15 new $(256, 85, 24, 30)$-SRGs in the following table. The notation $\mathtt{G}$ denotes the SRG, $\mathrm{Aut}(\mathtt{G})$ denotes the automorphism group of the graph $\mathtt{G}$, M denotes the adjacent matrix of $\mathtt{G}$, and $\mathrm{Rank}(M)$ denotes the 2-rank of the adjacent matrix M.

Table 1: New Negative Latin square type $(256, 85, 24, 30)$-SRGs from APN functions

No.	$\lvert\mathrm{Aut}(\mathtt{G})\rvert$	$\mathrm{Rank}(M)$	Remark	No.	$\lvert\mathrm{Aut}(\mathtt{G})\rvert$	$\mathrm{Rank}(M)$	Remark
1	2^9	256	new	2, 6	2^{11}	256	new
3	2^8	256	new	4	2^{10}	256	new
5	2^9	256	new	6	2^{11}	256	new
7	2^{10}	256	new	8	2^{10}	256	new
9	2^9	256	new	10	2^{10}	256	new
11	2^8	256	new	12	2^{10}	256	new
13, 14, 17	$2^{11} \cdot 5 \cdot 17$	256	Lemma 13	15	2^{10}	256	new
16	2^9	256	new	18	2^{10}	256	new

Finally, due to the relationship between quadratic zero-difference balanced function of the form $F(x) = G(x^{p^t+1})$ and SRGs, we leave the following problem for interested readers.

Problem 2. *Let t be a positive integer and $n \equiv 0 \mod 2t$, construct quadratic zero-difference p^t-balanced functions $F(x)$ of the form $G(x^{p^t+1})$ over $\mathbb{F}_{p^n}$, where the restriction of G to C_{p^t+1} is an injection. Particularly, find such functions when $n/2t$ is odd (as by Theorem 2 they can lead to negative Latin square type SRGs).*

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Appendix

Table 2: Quadratic APN functions on $\mathbb{F}_{2^8}$ with property P

No.	Function
1	$w^{132}x^{192} + w^{37}x^{144} + w^{91}x^{132} + w^{188}x^{129} + w^{76}x^{96} + w^{162}x^{72} + w^{46}x^{66} + w^{252}x^{48}$ $+w^{42}x^{36} + w^{81}x^{33} + w^{83}x^{24} + w^{13}x^{18} + w^{185}x^{12} + w^{163}x^{9} + w^{216}x^{6} + w^{181}x^{3}$
2	$x^{144} + x^6 + x^3$
3	$w^{91}x^{192} + w^{124}x^{144} + w^{214}x^{132} + w^{106}x^{129} + w^{59}x^{96} + w^{172}x^{72} + w^{138}x^{66} + w^{163}x^{48}$ $+w^{58}x^{36} + w^{100}x^{33} + w^{32}x^{24} + w^{250}x^{18} + w^{45}x^{12} + w^{241}x^{6} + w^{157}x^{3}$
4	$w^{21}x^{144} + w^{183}x^{66} + w^{245}x^{33} + x^3$
5	$w^{155}x^{192} + w^{96}x^{144} + w^{223}x^{132} + w^{77}x^{129} + w^{88}x^{96} + w^{232}x^{72} + w^{69}x^{66} + w^{142}x^{48}$ $+w^{168}x^{36} + x^{33} + w^{145}x^{24} + w^{234}x^{18} + w^{202}x^{12} + w^{94}x^{9} + w^{189}x^{6} + w^{241}x^{3}$
6	$x^{72} + x^6 + x^3$
7	$w^{126}x^{192} + w^{119}x^{144} + w^{221}x^{132} + w^{222}x^{129} + w^{79}x^{96} + w^{221}x^{72} + w^{187}x^{66}$ $+w^{148}x^{48} + w^{187}x^{36} + w^{237}x^{24} + w^{231}x^{12} + w^{119}x^{9} + w^{244}x^{6} + w^{236}x^{3}$
8	$w^{25}x^{192} + w^{140}x^{144} + w^{59}x^{132} + w^{129}x^{129} + w^{42}x^{96} + w^{164}x^{72} + w^{149}x^{66} + w^{119}x^{48}$ $+w^{74}x^{36} + w^{211}x^{33} + w^{9}x^{24} + w^{46}x^{18} + w^{130}x^{12} + w^{185}x^{9} + w^{147}x^{6} + w^{27}x^{3}$
9	$w^{151}x^{192} + w^{13}x^{144} + w^{58}x^{132} + w^{143}x^{129} + w^{110}x^{96} + wx^{72} + w^{244}x^{66} + w^{26}x^{48}$ $+w^{180}x^{36} + w^{8}x^{33} + w^{69}x^{24} + w^{76}x^{18} + w^{201}x^{12} + w^{201}x^{9} + w^{19}x^{6} + w^{107}x^{3}$
10	$w^{135}x^{144} + w^{120}x^{66} + w^{65}x^{18} + x^3$
11	$w^{113}x^{192} + w^{56}x^{144} + w^{68}x^{132} + w^{155}x^{129} + w^{91}x^{96} + w^{78}x^{72} + w^{159}x^{66}$ $+w^{30}x^{48} + w^{194}x^{36} + w^{14}x^{33} + w^{238}x^{24} + w^{91}x^{18} + w^{100}x^{12} + w^{96}x^{9} + w^{222}x^{6} + w^{178}x^{3}$
12	$w^{86}x^{192} + w^{224}x^{129} + w^{163}x^{96} + w^{102}x^{66} + w^{129}x^{48} + w^{102}x^{36} + w^{170}x^{33} + w^{14}x^{24}$ $+w^{170}x^{18} + w^{101}x^{12} + w^{58}x^{6} + w^{254}x^{3}$
13	x^9
14	x^3
15	$w^{95}x^{192} + w^{242}x^{144} + w^{195}x^{132} + w^{98}x^{129} + w^{84}x^{96} + w^{45}x^{72} + w^{234}x^{66}$ $+w^{202}x^{48} + w^{159}x^{36} + w^{58}x^{33} + w^{23}x^{24} + w^{148}x^{18} + w^{230}x^{12} + w^{32}x^{9} + w^{54}x^{6} + w^{41}x^{3}$
16	$w^{189}x^{192} + w^{143}x^{144} + w^{22}x^{132} + w^{21}x^{129} + w^{133}x^{96} + w^{239}x^{72} + w^{229}x^{66} + w^{31}x^{48}$ $+w^{187}x^{36} + w^{185}x^{33} + w^{68}x^{24} + w^{236}x^{18} + w^{75}x^{12} + w^{91}x^{9} + w^{97}x^{6} + w^{160}x^{3}$
17	x^{57}
18	$w^{67}x^{192} + w^{182}x^{132} + w^{24}x^{6} + x^3$