Nonincident Points and Blocks in Designs

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email from Doug West, August 24, 2011

How it started:

Hi Doug,
I have a question about projective planes that seems likely to be known. In a projective plane of order q, I want to take k points and k lines. How large does k have to be guarantee that some point in my set will lie in some line in my set, no matter which sets of size k I choose? I don't need the precise answer. Actually, $o(q^2)$ is probably good enough.
Best regards,
Doug

my reply, August 24, 2011

My initial reaction:

Hi Doug,
I'm not familiar with this question - maybe a geometer would be more likely to know the answer.
I did observe that you can construct counterexamples with k approximately equal to $q^{1.5}$ (which of course does not violate your conjecture). It is a fairly simple construction using maximal arcs (let me know if you want some details).
Doug.

The problem in projective planes

- Suppose $\Pi = (X, \mathcal{L})$ is a projective plane of order q, where X is the set of points and $\mathcal{L}$ is the set of lines in Π.
- For $Y \subseteq X$ and $\mathcal{M} \subseteq \mathcal{L}$, we say that $(Y, \mathcal{M})$ is a nonincident set of points and lines if $y \notin M$ for every $y \in Y$ and every $M \in \mathcal{M}$.
- Define $f(\Pi)$ to be the maximum integer s such that there exists a nonincident set of s points and s lines in Π.
- Equivalently, $f(\Pi)$ is the size of the largest square submatrix of zeroes in the incidence matrix of Π.

An example

$f(\mathsf{PG}(2,4)) \geq 6$:

3	**6**	7	12	14	10	13	14	19	0	**17**	20	**0**	5	7
4	**7**	**8**	**13**	**15**	**11**	**14**	**15**	**20**	**1**	18	**0**	1	**6**	8
5	**8**	**9**	**14**	**16**	12	15	16	**0**	**2**	**19**	1	**2**	7	9
6	9	10	15	**17**	13	16	**17**	1	**3**	20	**2**	**3**	8	10
7	10	11	16	18	14	**17**	18	**2**	4	**0**	**3**	4	9	11
8	11	12	**17**	**19**	15	18	**19**	**3**	5	**1**	**4**	**5**	**10**	**12**
9	**12**	**13**	**18**	**20**	**16**	**19**	**20**	**4**	**6**	**2**	5	**6**	11	13

$Y = \{\mathbf{0}, \mathbf{2}, \mathbf{3}, \mathbf{6}, \mathbf{17}, \mathbf{19}\}$.

A bound

Theorem

For any set Y *of* s *points in a projective plane* $\Pi = (X, \mathcal{L})$ *of order* q*, the number of lines disjoint from* Y *is at most*

$$\frac{q^3 + q^2 + q - qs}{q + s}.$$

Proof

- For a subset $Y \subseteq X$ of s points, let $\mathcal{B}$ consist of the nonempty intersections of the lines in $\mathcal{L}$ with the set Y.
- Denote $b = |\mathcal{B}|$.
- We have the following equations:

$$\begin{aligned} \sum_{B \in \mathcal{B}} 1 &= b \\ \sum_{B \in \mathcal{B}} |B| &= (q+1)s \\ \sum_{B \in \mathcal{B}} \binom{|B|}{2} &= \binom{s}{2}. \end{aligned}$$

- From the above equations, it follows that

$$\sum_{B \in \mathcal{B}} |B|^2 = s(q+s).$$

Proof (cont.)

- Define $\beta = (q+s)/(q+1)$ and compute as follows:

$$\begin{aligned} 0 &\leq \sum_{B \in \mathcal{B}} (|B| - \beta)^2 \\ &= s(q+s) - 2\beta(q+1)s + \beta^2 b, \end{aligned}$$

- It follows that

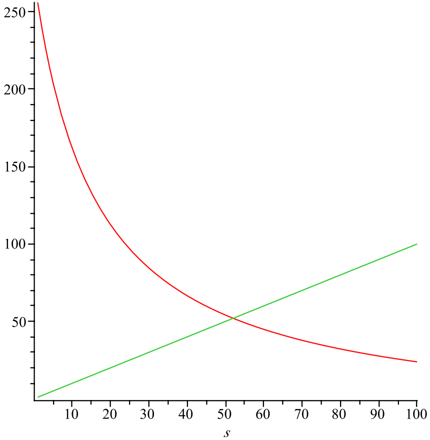
$$\begin{aligned} b &\geq \frac{s(2\beta(q+1) - (q+s))}{\beta^2} \\ &= \frac{(q+1)^2 s}{q+s}. \end{aligned}$$

- Therefore, the number of lines disjoint from Y is at most

$$q^2 + q + 1 - \frac{(q+1)^2 s}{q+s} = \frac{q^3 + q^2 + q - qs}{q+s}.$$

Example

We graph the functions $(q^3 + q^2 + q - qs)/(q + s)$ and s for $q = 16$ and $s \leq 100$. The point of intersection is $(52, 52)$ and therefore $f(\Pi) \leq 52$ for any projective plane Π of order 16.



The point of intersection

In general, it is easy to compute the point of intersection of these two functions as follows:

$$\frac{q^3+q^2+q-qs}{q+s} = s \quad \Leftrightarrow \quad s^2+2qs-(q^3+q^2+q)=0$$
$$\Leftrightarrow \quad s = -q \pm \sqrt{q^3+2q^2+q}$$
$$\Leftrightarrow \quad s = -q \pm (q+1)\sqrt{q}.$$

Since $s > 0$, the point of intersection occurs when

$$s = -q + (q+1)\sqrt{q} = 1 + (q+1)(\sqrt{q}-1).$$

The following result is now straightforward.

Theorem

For any projective plane Π of order q, it holds that $f(\Pi) \leq 1 + (q+1)(\sqrt{q}-1)$.

Maximal arcs

- A maximal (s, β)-arc in a projective plane of order q is a set Y of s points such that every line meets Y in 0 or β points.
- It is well-known that a maximal (s, β)-arc has $s = 1 + (q+1)(\beta - 1)$ and the number of lines that intersect the maximal arc is precisely

$$\frac{s(q+1)}{\beta} = \frac{s(q+1)^2}{q+s}.$$

- Denniston proved that there is a maximal $(s, 2^u)$-arc in $\mathrm{PG}(2, 2^v)$ whenever $0 < u < v$.
- In the case where q is odd, it was shown by Ball, Blokhuis and Mazzocca that there is no nontrivial maximal arc in the desarguesian plane $\mathrm{PG}(2, q)$. The existence of maximal arcs for nondesarguesian projective planes of odd order is unresolved at the present time.

When is equality attained in the bound?

- Suppose we have a set Y of s points in a projective plane of order q such that the number of lines disjoint from Y is equal to
$$\frac{q^3 + q^2 + q - qs}{q + s}.$$
- Then Y is a maximal (s, β)-arc, where $s = (q + 1)(\beta - 1) - 1$.
- Conversely, if Y is a maximal (s, β)-arc in a projective plane of order q, then number of lines disjoint from Y is equal to $(q^3 + q^2 + q - qs)/(q + s)$.

An infinite class of optimal solutions

Suppose $q = 2^v$ where v is even, and take $u = v/2$. There is a Denniston maximal (s, β)-arc, where $q = 2^v$, $\beta = \sqrt{q}$ and $s = 1 + (q+1)(\sqrt{q} - 1)$.

Theorem

If q is an even power of 2, then $f(PG(2, q)) = 1 + (q+1)(\sqrt{q} - 1)$.

Corollary

$f(PG(2, 4)) = 6$ and $f(PG(2, 16)) = 52$.

The analogous problem for Steiner triple systems

- A Steiner triple system of order v (or STS(v)), is a pair $(X, \mathcal{B})$, where X is the set of v points and $\mathcal{B}$ is a set of $b = v(v-1)/6$ blocks, such that each block contains three points and every pair of points occurs in a unique block.
- It is well-known that $v \equiv 1, 3 \bmod 6$ is a necessary and sufficient condition for the existence of an STS(v).
- Define $f(v)$ to be the maximum integer s such that there exists a nonincident set of s points and s blocks in some STS(v).
- As an example, consider an STS(9) that is a subdesign of an STS(21). There are 12 blocks in the subdesign and 12 points not in the subdesign.
- This implies that $f(21) \geq 12$.

A bound

We summarise the main results:

Theorem

For any set Y of s points in an STS(v), the number of blocks disjoint from Y is at most

$$\frac{v(v-1)+s^2-s(2v-1)}{6}.$$

Theorem

For any positive integer $v \equiv 1, 3 \bmod 6$, it holds that

$$f(v) \leq \frac{2v+5-\sqrt{24v+25}}{2}.$$

When is equality attained in the bound?

Theorem

Suppose that $(X, \mathcal{B})$ *is an STS*(v) *and suppose we have a set* $Y \subset X$ *of* s *points such that the number of blocks in* $\mathcal{B}$ *disjoint from* Y *is equal to*

$$\frac{v(v-1) + s^2 - s(2v-1)}{6}.$$

Then $(X \backslash Y, \mathcal{B}'_Y)$ *is a sub-STS*$(v-s)$ *of* $(X, \mathcal{B})$*, where* $\mathcal{B}'_Y$ *denotes the blocks in* $\mathcal{B}$ *that are disjoint from* Y*.*
Conversely, if $(Z, \mathcal{C})$ *is sub-STS*(w) *of* $(X, \mathcal{B})$*, then number of blocks in* $\mathcal{B}$ *disjoint from* $X \backslash Z$ *is equal to* $(v(v-1) + s^2 - s(2v-1))/6$*, where* $s = |X \backslash Z| = v - w$*.*

Necessary conditions

We require an STS(v) containing a sub-STS$(v-s)$, where $s = (2v + 5 - \sqrt{24v + 25})/2$.

Therefore, we need to determine the positive integers v such that the following conditions are satisfied:

1. $v \equiv 1, 3 \bmod 6$
2. $s = (2v + 5 - \sqrt{24v + 25})/2$ is an integer
3. $v - s \equiv 1, 3 \bmod 6$, and
4. $v \geq 2(v - s) + 1$.

The optimal solutions

Theorem (Doyen-Wilson Theorem)

There exists an STS(v) containing a sub-STS(w) if and only if $v \geq 2w + 1,\ v, w \equiv 1, 3 \bmod 6$.

Theorem

Suppose $v \equiv 1, 3 \bmod 6$ *is a positive integer. Then*

$$f(v) = \frac{2v + 5 - \sqrt{24v + 25}}{2}$$

if and only if $v = 216z^2 + 42z + 1,\ 216z^2 + 186z + 39,$ $216z^2 + 138z + 21$ *or* $216z^2 + 282z + 91$, *where* z *is a non-negative integer.*

The three smallest cases where $f(v)$ attains its optimal value are when $v = 21,\ s = 12$; $v = 39,\ s = 26$; and $v = 91,\ s = 70$.

References

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thank you for your attention!