Nonincident Points and Blocks in Designs

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Abstract

In this paper, we study the problem of finding the largest possible set of s points and s blocks in a balanced incomplete block design, such that that none of the s points lie on any of the s blocks. We investigate this problem for two types of BIBDs: projective planes and Steiner triple systems. For a projective plane of order q, we prove that $s \leq 1 + (q+1)(\sqrt{q} - 1)$ and we also show that equality can be attained in this bound whenever q is an even power of two. For a Steiner triple system on v points, we prove that $s \leq (2v + 5 - \sqrt{24v + 25})/2$ and we determine necessary and sufficient conditions for equality to be attained in this bound.

1 Introduction

A *finite linear space* is a pair $(X, \mathcal{B})$, where X is a set of *points* and $\mathcal{B}$ is a set of proper subsets of X called *blocks* such that every pair of points occurs in a unique block. If all blocks are of size k, then we have a $(v, k, 1)$-*balanced incomplete block design* (BIBD), where $v = |X|$. For $Y \subseteq X$ and $\mathcal{D} \subseteq \mathcal{B}$, we say that $(Y, \mathcal{D})$ is a *nonincident* set of points and blocks if $y \notin D$ for every $y \in Y$ and every $D \in \mathcal{D}$.

Erdös, Fowler, Sós and Wilson observed in [7] that if $(X, \mathcal{B})$ is a linear space on v points and Y consists of one point, then a nonincident set of points and blocks $(Y, \mathcal{D})$ has $|\mathcal{D}| \geq \lfloor v - \sqrt{v} \rfloor$. Metsch [8] studied a similar problem when $|Y| = d$, namely, finding a lower bound on the number of lines disjoint from d points.

In this paper, we investigate a complementary problem for certain classes of BIBDs. We are interested in nonincident sets $(Y, \mathcal{D})$ of points and blocks in BIBDs, where $|Y| = |\mathcal{D}| = s$, say, and s is as large as possible. This value of s is in fact the size of the largest square submatrix of zeroes in the incidence matrix of the design.

The two classes of BIBDs that we consider are projective planes and Steiner triple systems. The problem for projective planes has a connection with certain extremal problems in posets (see Choi, Milans and West [2]). For a projective plane of order q, we prove that $s \leq 1 + (q+1)(\sqrt{q} - 1)$ and we also show that equality can be attained in this bound whenever q is an even power of two. For a Steiner triple system on v points, we prove that $s \leq (2v + 5 - \sqrt{24v + 25})/2$ and we determine necessary and sufficient conditions for equality to be attained in this bound.

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2 Projective Planes

Suppose $\Pi = (X, \mathcal{L})$ is a projective plane of order q, i.e., a $(q^2+q+1, q+1, 1)$-BIBD. Here X is the set of points and $\mathcal{L}$ is the set of blocks (also called *lines*). Define $f(\Pi)$ to be the maximum integer s such that there exists a nonincident set of s points and s lines in Π.

Example 2.1. *We list the* 21 *lines in* $PG(2,4)$*. The six points in the set* $Y = \{\mathit{0}, \mathit{2}, \mathit{3}, \mathit{6}, \mathit{17}, \mathit{19}\}$ *are nonincident with the six lines whose points are presented in bold typeface. Therefore,* $f(PG(2,4)) \geq 6$ *(using results we will prove later in this section, it can be shown that* $f(PG(2,4)) = 6$*).*

3	*6*	7	12	14	10	13	14	19	0	*17*	20	*0*	5	7
4	**7**	**8**	**13**	**15**	**11**	**14**	**15**	**20**	**1**	18	*0*	1	*6*	8
5	**8**	**9**	**14**	**16**	12	15	16	*0*	*2*	*19*	1	*2*	7	9
6	9	10	15	*17*	13	16	*17*	1	*3*	20	*2*	*3*	8	10
7	10	11	16	18	14	*17*	18	*2*	4	*0*	*3*	4	9	11
8	11	12	*17*	*19*	15	18	*19*	*3*	5	**1**	**4**	**5**	**10**	**12**
9	**12**	**13**	**18**	**20**	**16**	**19**	**20**	**4**	**6**	*2*	5	*6*	11	13

We employ a simple combinatorial argument to prove the upper bound $f(\Pi) \leq 1+(q+1)(\sqrt{q}-1)$, which holds for any projective plane Π of order q. We also use certain maximal arcs to show that this bound is tight in certain cases, namely, for the desarguesian plane $\mathrm{PG}(2,q)$ when q is an even power of two.

Theorem 2.1. *For any set* Y *of* s *points in a projective plane of order* q*, the number of lines disjoint from* Y *is at most*

$$\frac{q^3+q^2+q-qs}{q+s}.$$

Proof. Suppose that $(X, \mathcal{L})$ is a projective plane of order q. For a subset $Y \subseteq X$ of s points, define $\mathcal{L}_Y = \{L \in \mathcal{L} : L \cap Y \neq \emptyset\}$ and define $\mathcal{L}'_Y = \mathcal{L} \setminus \mathcal{L}_Y$. Furthermore, for every $L \in \mathcal{L}_Y$, define $L_Y = L \cap Y$, and then define $\mathcal{B} = \{L_Y : L \in \mathcal{L}_Y\}$. Observe that $\mathcal{B}$ consists of the nonempty intersections of the lines in $\mathcal{L}$ with the set Y. Denote $b = |\mathcal{B}| = |\mathcal{L}_Y|$.

We will study the set system $(Y, \mathcal{B})$. We have the following equations:

$$\begin{aligned} \sum_{B \in \mathcal{B}} 1 &= b \\ \sum_{B \in \mathcal{B}} |B| &= (q+1)s \\ \sum_{B \in \mathcal{B}} \binom{|B|}{2} &= \binom{s}{2}. \end{aligned}$$

From the above equations, it follows that

$$\sum_{B \in \mathcal{B}} |B|^2 = s(q+s).$$

The $q+1$ blocks in $\mathcal{B}$ through any point $y \in Y$ contain the $s-1$ points in $Y \setminus \{y\}$ once each, as well as $q+1$ occurrences of y. So the average size of a block in $\mathcal{B}$ that contains any given point $y \in Y$ is $(q+s)/(q+1)$. Therefore we define $\beta = (q+s)/(q+1)$ and compute as follows:

$$\begin{aligned} 0 &\leq \sum_{B \in \mathcal{B}} (|B| - \beta)^2 \\ &= s(q+s) - 2\beta(q+1)s + \beta^2 b, \end{aligned}$$

from which it follows that

$$\begin{aligned} b &\geq \frac{s(2\beta(q+1) - (q+s))}{\beta^2} \\ &= \frac{(q+1)^2 s}{q+s}. \end{aligned}$$

Therefore,

$$\begin{aligned} |\mathcal{L}'_Y| &= q^2 + q + 1 - b \\ &\leq q^2 + q + 1 - \frac{(q+1)^2 s}{q+s} \\ &= \frac{q^3 + q^2 + q - qs}{q+s}. \end{aligned}$$

□

Remark. The inequality $b \geq (q+1)^2 s/(q+s)$ is actually a well-known result that has been proven in many different guises over the years. For example, Mullin and Vanstone [10] proved that $b \geq r^2 v/(r + \lambda(v-1))$ in any (r, λ) design on v points. If we let $r = q+1$, $\lambda = 1$ and $v = s$, then we obtain $b \geq (q+1)^2 s/(q+s)$.

Corollary 2.2. *If there exists a nonincident set of s points and t lines in a projective plane of order q, then*

$$t \leq \frac{q^3 + q^2 + q - qs}{q+s}.$$

Before proving our next general result, we look at a small example. In Figure 1, we graph the functions $(q^3 + q^2 + q - qs)/(q+s)$ and s for $q = 16$ and $s \leq 100$. The point of intersection is $(52, 52)$ and it is then easy to see that $f(\Pi) \leq 52$ for any projective plane Π of order 16.

In general, it is easy to compute the point of intersection of these two functions as follows:

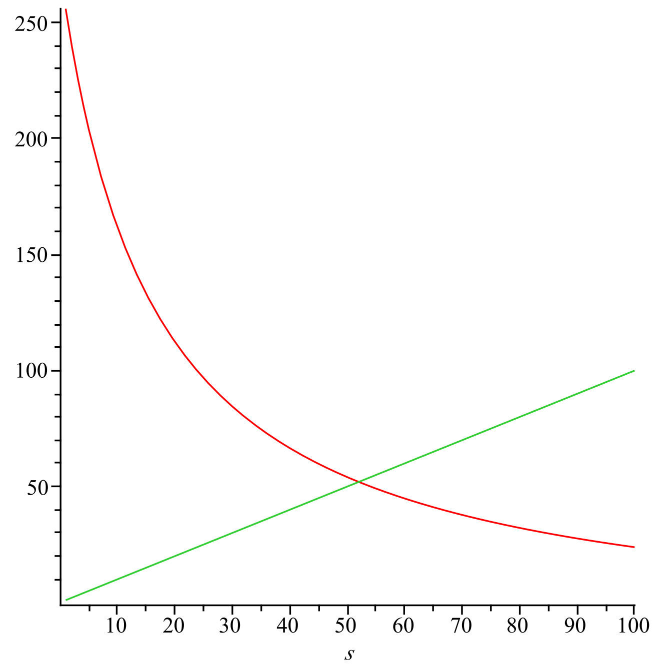
$$\begin{aligned} \frac{q^3 + q^2 + q - qs}{q+s} = s \quad &\Leftrightarrow \quad s^2 + 2qs - (q^3 + q^2 + q) = 0 \\ &\Leftrightarrow \quad s = -q \pm \sqrt{q^3 + 2q^2 + q} \\ &\Leftrightarrow \quad s = -q \pm (q+1)\sqrt{q}. \end{aligned}$$

Since $s > 0$, the point of intersection occurs when

$$s = -q + (q+1)\sqrt{q} = 1 + (q+1)(\sqrt{q} - 1).$$

The following result is now straightforward.

Figure 1: Nonincident points and lines when $q = 16$



Theorem 2.3. *For any projective plane* Π *of order* q*, it holds that* $f(\Pi) \leq 1 + (q+1)(\sqrt{q} - 1)$.

Proof. Suppose there is a nonincident set of s points and s lines in a projective plane of order q. Theorem 2.1 implies that $s \leq (q^3 + q^2 + q - qs)/(q + s)$. However, for $s > 1 + (q+1)(\sqrt{q} - 1)$, we have that $s > (q^3 + q^2 + q - qs)/(q + s)$, just as in the example considered above. It follows that $s \leq 1 + (q+1)(\sqrt{q} - 1)$. □

Next, we examine the case of equality in Theorem 2.1. This will involve maximal arcs, which we now define. A *maximal* (s, β)*-arc* in a projective plane of order q is a set Y of s points such that every line meets Y in 0 or β points. It is well-known that a maximal (s, β)-arc has $s = 1 + (q+1)(\beta - 1)$ points and the number of lines that intersect the maximal arc is precisely

$$\frac{s(q+1)}{\beta} = \frac{s(q+1)^2}{q+s}.$$

A projective plane of order q always has a *trivial* maximal $(q^2 + q + 1, q + 1)$-arc consisting of all the points. It is easy to show that $\beta \mid q$ for any nontrivial maximal arc. For additional information on maximal arcs, see [3, §VI.41.3].

Corollary 2.4. *Suppose we have a set* Y *of* s *points in a projective plane of order* q *such that the number of lines disjoint from* Y *is equal to*

$$\frac{q^3 + q^2 + q - qs}{q + s}.$$

Then Y is a maximal (s,β)-arc, where $s=(q+1)(\beta-1)+1$. Conversely, if Y is a maximal (s,β)-arc in a projective plane of order q, then number of lines disjoint from Y is equal to $(q^3+q^2+q-qs)/(q+s)$.

Proof. From the proof of Theorem 2.1, it is easy to see that equality holds if and only if every line in $\mathcal{L}_Y$ meets Y in exactly β points, where $\beta=(q+s)/(q+1)$. It immediately follows that every line in the plane meets Y in 0 or β points, and therefore Y is a maximal arc. The converse follows from the basic properties of maximal arcs mentioned above. □

Theorem 2.5. *When q is an even power of 2, there exist nonincident sets of s points and s lines in $PG(2,q)$, where $s=1+(q+1)(\sqrt{q}-1)$.*

Proof. Denniston [5] proved that there is a maximal $(s,2^u)$-arc in $\mathrm{PG}(2,2^v)$ whenever $0<u<v$. Suppose v is even and we take $u=v/2$. Therefore we have a maximal (s,β)-arc, where $q=2^v$, $\beta=\sqrt{q}$ and $s=1+(q+1)(\sqrt{q}-1)$.

Suppose we take Y to be the s points in the arc and we apply Corollary 2.4. Since $s=1+(q+1)(\beta-1)$, there are exactly s lines in $\mathcal{L}'_Y$. Therefore we have a nonincident set of s points and s lines in $\mathrm{PG}(2,q)$. □

Corollary 2.6. *If q is an even power of 2, then $f(PG(2,q))=1+(q+1)(\sqrt{q}-1)$.*

Proof. From Theorem 2.5, we have $f(\mathrm{PG}(2,q))\geq 1+(q+1)(\sqrt{q}-1)$. However, $f(\mathrm{PG}(2,q))\leq 1+(q+1)(\sqrt{q}-1)$ from Theorem 2.3. It follows that $f(\mathrm{PG}(2,q))=1+(q+1)(\sqrt{q}-1)$. □

Remark. The construction given in Example 2.1 is an application of Theorem 2.5. In this case, the relevant maximal arc is just a hyperoval in $\mathrm{PG}(2,4)$.

In the case where q is odd, it was shown by Ball, Blokhuis and Mazzocca [1] that there is no nontrivial maximal arc in the desarguesian plane $\mathrm{PG}(2,q)$. The existence of nontrivial maximal arcs for nondesarguesian projective planes of odd order is unresolved at the present time. For the problem we are considering, an optimal solution would correspond to an $(s,\sqrt{q})$-arc in a nondesarguesian projective plane of order q, where q is (necessarily) a perfect square.

3 Steiner Triple Systems

A *Steiner triple system of order v* (or $\mathrm{STS}(v)$), is a pair $(X,\mathcal{B})$, where X is the set of v *points* and $\mathcal{B}$ is a set of $b=v(v-1)/6$ *blocks*, such that each block contains three points and every pair of points occurs in a unique block. An $\mathrm{STS}(v)$ is the same thing as a $(v,3,1)$-BIBD. It is well-known that $v\equiv 1,3 \bmod 6$ is a necessary and sufficient condition for the existence of an $\mathrm{STS}(v)$.

A *maximal arc* in an $\mathrm{STS}(v)$ consists of a subset Y of $(v+1)/2$ points such that every block meets Y in 0 or 2 points (see, for example, [9, Theorem 3.1]). When $v\equiv 3,7 \bmod 12$, $\mathrm{STS}(v)$ containing maximal arcs can easily be constructed using the standard "doubling construction" (e.g., see [4, §3.2]). The number of blocks disjoint from Y is

$$\frac{v(v-1)}{6}-\binom{(v+1)/2}{2}=\frac{v^2-4v+3}{24}.$$

For $v\equiv 3,7 \bmod 12$, $v\geq 19$, it is easy to see that $(v^2-4v+3)/24>(v+1)/2$. For these values of v, this implies that we can find $s=(v+1)/2$ points and nonincident blocks in an $\mathrm{STS}(v)$ that contains a maximal arc. However, it turns out that we can do better, and we will show that the optimal value of s is roughly $v - \sqrt{6v}$, for infinitely many values of v.

Define $f_{\mathrm{STS}}(v)$ to be the maximum integer s such that there exists a nonincident set of s points and s blocks in some STS(v). We prove the upper bound $f_{\mathrm{STS}}(v) \leq (2v + 5 - \sqrt{24v+25})/2$. We also show that this bound is tight for infinitely many values of v.

Theorem 3.1. *For any set Y of s points in an STS(v), the number of blocks disjoint from Y is at most*

$$\frac{v(v-1) + s^2 - s(2v-1)}{6}.$$

Proof. Suppose that $(X, \mathcal{B})$ is an STS(v). Denote $r = (v-1)/2$; then every point occurs in r blocks in the STS(v). For a subset $Y \subseteq X$ of s points, define $\mathcal{B}_Y = \{B \in \mathcal{B} : B \cap Y \neq \emptyset\}$ and define $\mathcal{B}'_Y = \mathcal{B} \setminus \mathcal{B}_Y$. Furthermore, for every $B \in \mathcal{B}_Y$, define $B_Y = B \cap Y$, and then define $\mathcal{C} = \{B_Y : B \in \mathcal{B}_Y\}$. Observe that $\mathcal{C}$ consists of the nonempty intersections of the blocks in $\mathcal{B}$ with the set Y. Denote $c = |\mathcal{C}| = |\mathcal{B}_Y|$.

We will study the set system $(Y, \mathcal{C})$. We have the following equations:

$$\begin{aligned}
\sum_{C \in \mathcal{C}} 1 &= c \\
\sum_{C \in \mathcal{C}} |C| &= rs \\
\sum_{C \in \mathcal{C}} \binom{|C|}{2} &= \binom{s}{2}.
\end{aligned}$$

From the above equations, it follows that

$$\sum_{C \in \mathcal{C}} |C|^2 = s(s + r - 1).$$

Now we compute as follows:

$$\begin{aligned}
0 &\leq \sum_{C \in \mathcal{C}} (|C| - 2)(|C| - 3) \\
&= s(s + r - 1) - 5rs + 6c,
\end{aligned}$$

from which it follows that

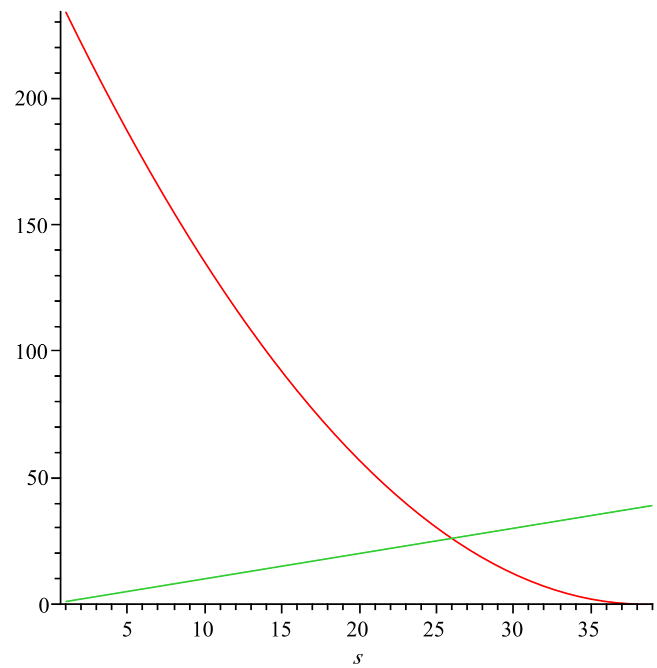
$$\begin{aligned}
c &\geq \frac{s(4r+1) - s^2)}{6} \\
&= \frac{s(2v-1) - s^2)}{6}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\mathcal{B}'_Y| &= b - c \\
&\leq \frac{v(v-1)}{6} - \frac{s(2v-1) - s^2)}{6} \\
&= \frac{v(v-1) + s^2 - s(2v-1)}{6}.
\end{aligned}$$

□

Figure 2: Nonincident points and lines when $v = 39$



Corollary 3.2. *If there exists a nonincident set of s points and t blocks in an $STS(v)$, then*

$$t \leq \frac{v(v-1) + s^2 - s(2v-1)}{6}.$$

Before proving our next general result, we look at a small example. In Figure 2, we graph the functions $(v(v-1) + s^2 - s(2v-1))/6$ and s for $v = 39$ and $s \leq v$. The point of intersection is $(26, 26)$ and it is then easy to see that $f(39) \leq 26$.

In general, it is easy to compute the point of intersection of these two functions as follows:

$$\begin{aligned} \frac{v(v-1) + s^2 - s(2v-1)}{6} = s \quad &\Leftrightarrow \quad s^2 - (2v+5)s + v^2 - v = 0 \\ &\Leftrightarrow \quad s = \frac{2v + 5 \pm \sqrt{24v + 25}}{2}. \end{aligned}$$

Since $s < v$, the point of intersection occurs when

$$s = \frac{2v + 5 - \sqrt{24v + 25}}{2}.$$

The following result is now straightforward.

Theorem 3.3. *For any positive integer $v \equiv 1, 3 \bmod 6$, it holds that $f_{\mathrm{STS}}(v) \leq \frac{2v+5-\sqrt{24v+25}}{2}$.*

Proof. Suppose there is a nonincident set of s points and s blocks in an STS(v). Theorem 3.1 implies that $s \leq (v(v-1)+s^2-s(2v-1))/6$. However, for $s > (2v+5-\sqrt{24v+25})/2$, we have that $s > (v(v-1)+s^2-s(2v-1))/6$, just as in the example considered above. It follows that $s \leq (2v+5-\sqrt{24v+25})/2$. □

Next, we examine the case of equality in Theorem 3.1. This will involve *subdesigns* of STS(v), which we now define. Suppose that $(X, \mathcal{B})$ is an STS(v). We say that $(Z, \mathcal{D})$ is a *sub-STS*(w) of $(X, \mathcal{B})$ if $Z \subset X$, $\mathcal{D} \subset \mathcal{B}$ and $(Z, \mathcal{D})$ is an STS(w). It is easy to see that an STS(v) containing a sub-STS(w) can exist only if $v \geq 2w+1$.

Corollary 3.4. *Suppose that $(X, \mathcal{B})$ is an STS(v) and suppose we have a set $Y \subset X$ of s points such that the number of blocks in $\mathcal{B}$ disjoint from Y is equal to*

$$\frac{v(v-1)+s^2-s(2v-1)}{6}.$$

Then $(X \backslash Y, \mathcal{B}'_Y)$ is a sub-STS($v-s$) of $(X, \mathcal{B})$, where $\mathcal{B}'_Y$ denotes the blocks in $\mathcal{B}$ that are disjoint from Y.

Conversely, if $(Z, \mathcal{C})$ is sub-STS(w) of $(X, \mathcal{B})$, then number of blocks in $\mathcal{B}$ disjoint from $X \backslash Z$ is equal to $(v(v-1)+s^2-s(2v-1))/6$, where $s = |X \backslash Z| = v - w$.

Proof. From the proof of Theorem 3.1, it is easy to see that equality holds if and only if every block in $\mathcal{B}_Y$ meets Y in exactly 2 or 3 points. This means that no block contains two points of $X \backslash Y$, and hence $(X \backslash Y, \mathcal{B}'_Y)$ is a sub-STS($v-s$) of $(X, \mathcal{B})$

Conversely, suppose that $(Z, \mathcal{C})$ is a sub-STS(w) of $(X, \mathcal{B})$. $\mathcal{C}$ consists of $w(w-1)/6$ blocks, and there are $v-w$ points in $X \backslash Z$. The blocks in $\mathcal{C}$ are all disjoint from $X \backslash Z$, so it suffices to verify that

$$\frac{v(v-1)+(v-w)^2-(v-w)(2v-1)}{6} = \frac{w(w-1)}{6}.$$

This is an easy computation. □

As an example, consider an STS(9) that is a subdesign of an STS(21). There are 12 blocks in the subdesign and 12 points not in the subdesign, which implies that $f(21) \geq 12$.

Our goal is to determine the integers v such that $f(v) = (2v+5-\sqrt{24v+25})/2$, i.e., where the bound in Theorem 3.3 is met with equality. In view of Corollary 3.4, this can happen if and only if there is an STS(v) containing a sub-STS($v-s$), where $s = (2v+5-\sqrt{24v+25})/2$. Therefore, we next determine the integers v such that the following conditions are satisfied:

1. $v \equiv 1, 3 \bmod 6$
2. $s = (2v+5-\sqrt{24v+25})/2$ is an integer
3. $v - s \equiv 1, 3 \bmod 6$, and
4. $v \geq 2(v-s)+1$.

First, condition 2 implies that $24v+25$ is a perfect square, say $24v+25 = t^2$. Then we have

$$v = \frac{t^2-25}{24} \quad \text{and} \quad s = v - \frac{t-5}{2}.$$

Observe that v is an integer only when $t \equiv 1, 5 \bmod 6$.

First, suppose $t \equiv 1 \bmod 6$ and write $t = 6u + 1$. It is then easy to see that

$$v = \frac{3u^2 + u - 2}{2} \quad \text{and} \quad s = \frac{3u^2 - 5u + 2}{2}.$$

Now, we consider requirements 1 and 3. A straightforward calculation shows that these conditions are satisfied if and only if $u \equiv 1, 5 \bmod 12$. If we let $u = 12z + 1$, then we get

$$v = 216z^2 + 42z + 1 \quad \text{and} \quad s = 216z^2 + 6z, \tag{1}$$

while if $u = 12z + 5$, we have

$$v = 216z^2 + 186z + 39 \quad \text{and} \quad s = 216z^2 + 150z + 26. \tag{2}$$

The case $t \equiv 5 \bmod 6$ is handled in a similar way. We can write $t = 6u - 1$ and then we compute

$$v = \frac{3u^2 - u - 2}{2} \quad \text{and} \quad s = \frac{3u^2 - 7u + 4}{2}.$$

Here it turns out that requirements 1 and 3 are satisfied if and only if $u \equiv 4, 8 \bmod 12$. If we let $u = 12z + 4$, then we get

$$v = 216z^2 + 138z + 21 \quad \text{and} \quad s = 216z^2 + 102z + 12, \tag{3}$$

while if $u = 12z + 8$, we have

$$v = 216z^2 + 282z + 91 \quad \text{and} \quad s = 216z^2 + 246z + 70. \tag{4}$$

In all four cases (1), (2), (3) and (4), it is easy to see that condition 4 is automatically satisfied. It follows that these four cases are the only situations where it is possible to have equality in Theorem 3.3. We now show that the desired designs exist in these cases, by making use of the following well-known result first proven in [6].

Theorem 3.5 (Doyen-Wilson Theorem)**.** *There exists an STS*(v) *containing a sub-STS*(w) *if and only if* $v \geq 2w + 1$*,* $v, w \equiv 1, 3 \bmod 6$*.*

The above discussion and Theorem 3.5 immediately imply our main existence result.

Theorem 3.6. *Suppose* $v \equiv 1, 3 \bmod 6$ *is a positive integer. Then* $f_{\mathrm{STS}}(v) = \frac{2v+5-\sqrt{24v+25}}{2}$ *if and only if*

$$v \in \{216z^2 + 42z + 1, 216z^2 + 186z + 39, 216z^2 + 138z + 21, 216z^2 + 282z + 91\},$$

where z *is a non-negative integer.*

The three smallest cases where $f(v)$ attains its optimal value are when $v = 21$, $s = 12$; $v = 39$, $s = 26$; and $v = 91$, $s = 70$.

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