Fault tolerant routings with minimum optical index

Jeffrey H. Dinitz and Alan Ling
University of Vermont

Douglas R. Stinson
University of Waterloo
Routings

- Let $n \geq 3$ be a positive integer and let $\vec{K}_n$ denote the complete directed graph on a set of $n$ vertices, say $X$.
- A $P_2$-routing of $\vec{K}_n$ is a set $\mathcal{L}$ of directed paths of length two in $\vec{K}_n$, such that the following properties are satisfied:
  1. for all $x, y \in X$, $x \neq y$, there is a unique directed path in $\mathcal{L}$ having origin $x$ and terminus $y$, and
  2. every arc in $\vec{K}_n$ occurs in exactly two directed paths in $\mathcal{L}$.
**Levelled Routings**

- A **levelled routing of** $\vec{K}_n$ is an $(n - 1)$-tuple $\mathcal{R} = (\mathcal{L}_0, \ldots, \mathcal{L}_{n-2})$ that satisfies the following properties:
  
  1. $\mathcal{L}_0$ consists of the $n(n - 1)$ arcs in $\vec{K}_n$ (i.e., all the possible directed paths of length 1),
  2. for $1 \leq i \leq n - 2$, $\mathcal{L}_i$ is a $P_2$-routing of $\vec{K}_n$, and
  3. every directed path of length two occurs in exactly one of the routings $\mathcal{L}_i$, $1 \leq i \leq n - 2$.

- The $\mathcal{L}_i$’s are called the **levels** of the routing.

- Levelled routings are of interest due to their fault-tolerant properties.
Optical Index

- In optical networks, we want to assign a **wavelength** to each directed path in \( \mathcal{R} \) in such a way that two directed paths can be assigned the same wavelength only if they do not contain a common arc.

- The **optical index** of \( \mathcal{R} \), denoted \( \mathcal{\bar{w}}(\mathcal{R}) \), is the minimum number of wavelengths in such a wavelength assignment.

- The **levelled optical index** of \( \mathcal{R} \), denoted \( \mathcal{\bar{w}}_L(\mathcal{R}) \), is the minimum number of wavelengths in such a wavelength assignment, satisfying the additional property that directed paths in different levels are always assigned different wavelengths.
Example

The (unique) levelled routing of $K_3^2$ is as follows:

\[
\mathcal{L}_0 : \ 01, 02, 10, 12, 20, 21 \\
\mathcal{L}_1 : \ 021, 012, 120, 102, 210, 201
\]

Note that we are using the notation $xy$ to denote the arc $x \to y$, and $xyz$ denotes the directed path $x \to y \to z$.

This routing has optical index 3:

\[
\mathcal{L}_0 : \ 01^3, 02^2, 10^1, 12^2, 20^1, 21^3 \\
\mathcal{L}_1 : \ 021^1, 012^1, 120^3, 102^3, 210^2, 201^2
\]

However, its levelled optical index is 4.
Bounds on the Optical Index

- The **path graph** of $\mathcal{L}_i$ is the graph whose vertices are the directed paths in $\mathcal{L}_i$.

- Two vertices (i.e., directed paths) are joined by an edge iff they have a common arc.

- Let $\delta_i$ denote the chromatic number of the path graph of $\mathcal{L}_i$.

- The path graph of $\mathcal{L}_i$ is a union of disjoint cycles, so $2 \leq \delta_i \leq 3$ for all $i, 1 \leq i \leq n - 2$.

- Also, $\delta_0 = 1$.

**Theorem 1 [Gupta, Maňuch and Stacho, 2006]**

For any levelled routing $\mathcal{R}$, it holds that

$$2n - 3 \leq \bar{\omega}(\mathcal{R}) \leq \bar{\omega}_L(\mathcal{R}) \leq 3n - 5.$$
Example

A levelled routing $\mathcal{R}$ of $\overrightarrow{K}_4$ with (optimal) levelled optical index $
abla_\mathcal{L} (\mathcal{R}) = 5$:

$$\mathcal{L}_0 = \{01, 02, 03, 10, 12, 13, 20, 21, 23, 30, 31, 32\}$$
$$\mathcal{L}_1 = \{031, 032, 013, 120, 102, 123, 230, 201, 213, 310, 321, 302\}$$
$$\mathcal{L}_2 = \{130, 132, 103, 021, 012, 023, 231, 210, 203, 301, 320, 312\}$$

The path graphs of $\mathcal{L}_1$ and $\mathcal{L}_2$ are both 12-cycles.
A New Recursive Construction

- A $t-(v, k, \lambda)$-design $(X, \mathcal{A})$ is said to be partitionable if it is possible to partition $\mathcal{A}$ into

$$\ell = \frac{v - t + 1}{k - t + 1}$$

subcollections $\mathcal{A}_i$ ($1 \leq i \leq \ell$) such that $(X, \mathcal{A}_i)$ is a $(t - 1)-(v, k, \lambda)$-design for all $i$, $1 \leq i \leq \ell$.

- If there exists a partitionable 3-$(n, 4, 1)$-design, then there exists a levelled routing of $\vec{K}_n$ that has (optimal) levelled optical index $\omega_L(\mathcal{R}) = 2n - 5$.

- Baker and Teirlinck proved for all $n = 4^k$, $n = 2(7^k + 1)$ and $n = 2(31^k + 1)$ that there exists a partitionable 3-$(n, 4, 1)$-design.

- Hence, for these values of $n$, there exist levelled routings of $\vec{K}_n$ that have optimal levelled optical indices.
Levelled Routings for all $n$

- Two idempotent latin squares of order $n$, say $L_1$ and $L_2$, defined on the same symbol set $X$, are **disjoint** if $L_1(x, y) \neq L_2(x, y)$ for all $x \neq y$.

- $L_1, \ldots, L_f$ are disjoint idempotent latin squares of order $n$ if every pair of squares $L_i, L_j$ ($i \neq j$) are disjoint idempotent latin squares of order $n$.

- If $L_1, \ldots, L_f$ are disjoint idempotent latin squares of order $n$, then $f \leq n - 2$.

- Teirlinck, Lindner and Chang proved that there exist $n - 2$ disjoint idempotent latin squares of order $n$ for all $n \neq 6$. 
Levelled Routings for all $n$ (cont.)

**Theorem 2 [Gupta, Maňuch and Stacho, 2006]**

Suppose $L_1, \ldots, L_{n-2}$ are disjoint idempotent latin squares of order $n$. For $1 \leq i \leq n - 2$, define

$$L_i = \{xyz : z = L_i(x, y), x \neq y\}.$$

Also, define

$$L_0 = \{xy : x \neq y\}.$$

Then $(L_0, \ldots, L_{n-2})$ is a levelled routing of $\vec{K}_n$.

- Hence, there exist levelled routings of $\vec{K}_n$ for all $n \neq 6$.
- Furthermore, we constructed a levelled routing of $\vec{K}_6$, so levelled routings of $\vec{K}_n$ exist for all $n \neq 2$. 
An Algebraic Construction

- Suppose that \( q \) is an odd prime power and let \( X = \mathbb{F}_q \).

- For all \( a \in \mathbb{F}_q, a \neq 0, 1 \), define the latin square \( L_a \) by the formula
  \[
  L_a(x, y) = ax + (1 - a)y.
  \]

- Then \( L_a \) is an idempotent latin square of order \( q \) for all \( a \in \mathbb{F}_q, a \neq 0, 1 \).

- Further, \( L_a \) and \( L_b \) are disjoint if \( a \neq b \).

- Hence, we obtain a levelled routing, say \( \mathcal{R} \), of \( \vec{K}_q \).
Chromatic Numbers of the Path Graphs

- The routing paths in $L_a$ are $xyz$, $z = ax + (1 - a)y$, $x \neq y$.
- We can express $y$ as a function of $x$ and $z$ as follows:
  \[ y = \frac{z - ax}{1 - a}. \]
- Thus we have
  \[ L_a = \{ P_{x,z} = xyz : y = (z - ax)(1 - a)^{-1}, x \neq z \}. \]
- Consider the recurrence relation
  \[ x_{i+2} = \frac{x_{i+1} - ax_i}{1 - a}, \]
  \[ i = 0, 1, \ldots. \]
Solution of the Recurrence Relation

- Suppose that $a \in \mathbb{F}_q$, $a \neq 0, 1, 2^{-1}$.
- Then (1) has the solution
  
  $$x_i = c_0 + c_1 \gamma^i,$$
  
  where $\gamma = a(1 - a)^{-1}$, $c_0 = (x_1 - \gamma x_0)(1 - \gamma)^{-1}$ and $c_1 = (x_0 - x_1)(1 - \gamma)^{-1}$.
- If $a = 2^{-1}$, then (1) has the solution
  
  $$x_i = x_0 + i(x_1 - x_0).$$

Theorem 3 [DLS]

Suppose that $a \in \mathbb{F}_q$, $a \neq 0, 1, 2^{-1}$. Then the path graph of $\mathcal{L}_a$ consists of disjoint cycles of order $d$, where $d$ is the (multiplicative) order of $a/(1 - a)$ in $\mathbb{F}_q^*$. If $a = 2^{-1}$, then the path graph of $\mathcal{L}_a$ consists of disjoint cycles of order $q$. 
**Example: $q = 7$**

- Here, $2^{-1} = 4$, and the path graph of $\mathcal{L}_4$ consists of six disjoint cycles of order 7.

- When $a = 2, 3, 5, 6$, it can be verified that $a/(1 - a) = 5, 2, 4, 3$ (respectively).

- The field elements 2 and 4 have order 3, and 3 and 5 have order 6 in $\mathbb{F}_7^*$. 

- Therefore $\mathcal{L}_2$ and $\mathcal{L}_6$ each consist of seven disjoint cycles of order 6, while $\mathcal{L}_3$ and $\mathcal{L}_5$ each consist of fourteen disjoint cycles of order 3.
Example: $q = 7$ (cont.)

Let’s look at $\mathcal{L}_2$ in more detail. If we take $x_0 = 0, x_1 = 1$, then we obtain $x_2 = 6, x_3 = 3, x_4 = 2, x_5 = 4$, and $x_{i+6} = x_i$ for all $i \geq 0$. The following six paths form a cycle in the path graph of $\mathcal{L}_2$:

$016 \rightarrow 163 \rightarrow 632 \rightarrow 324 \rightarrow 240 \rightarrow 401 \rightarrow 016$. The seven 6-cycles in this path graph are as follows:

$$
016 - 163 - 632 - 324 - 240 - 401 - 016 \\
120 - 204 - 043 - 435 - 351 - 512 - 120 \\
231 - 315 - 154 - 546 - 462 - 623 - 231 \\
342 - 426 - 265 - 650 - 503 - 034 - 342 \\
453 - 530 - 306 - 061 - 614 - 145 - 453 \\
564 - 641 - 410 - 102 - 025 - 256 - 564 \\
$$
Levelled Optical Index

- Suppose $q$ is an odd prime power and write $q - 1 = 2^s t$, where $t$ is odd.
- if $a = 2^{-1}$, then $a/(1 - a) = 1$ and the path graph of $\mathcal{L}_a$ has chromatic number equal to 3,
- there are $t - 1$ elements $a \neq 2^{-1}$ such that $a/(1 - a)$ has odd order (the path graphs of these $\mathcal{L}_a$'s have chromatic number equal to 3), and
- for the remaining $q - 2 - t$ values of $a$, the path graphs of the $\mathcal{L}_a$’s have chromatic number equal to 2.

**Theorem 4 [DLS]**

*Suppose $q$ is an odd prime power and write $q - 1 = 2^s t$, where $t$ is odd. Then there is a levelled routing $\mathcal{R}_{q-2}$ of $K_2^q$ such that $\overline{w}_L(\mathcal{R}) = 2q - 3 + t.*
Levelled vs Unlevelled Optical Index

● In the “worst” case, when \( s = 1 \), we have \( t = (q - 1)/2 \) and 
\[ \bar{w}_L(\mathcal{R}) = (5q - 1)/2. \]

● This can (sometimes) be improved if we do not require levelled optical indices.

Theorem 5 [DLS]
Suppose that \( q = 2p + 1 \) is prime, where \( p \) is an odd prime. Then there exists a levelled routing \( \mathcal{R} \) of \( \overrightarrow{K}_q \) such that \( \bar{w}(\mathcal{R}) \leq 2q + 6. \)
References

