On Randomizing Private Keys to Counteract DPA Attacks

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Abstract

Differential power analysis (DPA) attacks can be of major concern when applied to cryptosystems that are embedded into small devices such as smart cards. To immunize elliptic curve cryptosystems (ECCs) against DPA attacks, recently several countermeasures have been proposed. A class of countermeasures is based on randomizing the paths taken by the scalar multiplication algorithm throughout its execution which also implies generating a random binary signed-digit (BSD) representation of the scalar. This scalar is an integer and is the secret key of the cryptosystem. In this report, we investigate issues related to the BSD representation of an integer such as the average and the exact number of these representations, and integers with maximum number of BSD representations within a specific range. This knowledge helps a cryptographer to choose a key that provides better resistance against DPA attacks. Here, we also present an algorithm that generates a random BSD representation of an integer starting from the most significant signed bit. We also present another algorithm that generates all existing BSD representations of an integer to investigate the relation between increasing the number of bits in which an integer is represented and the increase in the number of its BSD representations.

Keywords: Differential power analysis, elliptic curve cryptosystems, binary signed-digit representation, scalar multiplication, smart cards.

1 Introduction

Elliptic curve cryptosystems (ECC) have gained increased popularity since they were first proposed by Miller [17] and Koblitz [9]. Elliptic curves (EC) over large finite fields are governed by a group law that makes the discrete logarithm problem over that group much harder to solve than over other groups such as the multiplicative group of a finite field. This has led to employing much smaller size keys in ECCs compared to RSA cryptosystems. Hence, better performance in computation speed can be obtained. These advantages make ECCs more attractive for cryptographic implementation on smart cards since they are constrained with limited memory and computation capability. This is also the case for wireless devices which have limited power as well.

However, smart cards and wireless devices are vulnerable to a special type of attacks, those are side-channel attacks. For these systems, a correct implementation of a strong protocol is not necessarily secure if this implementation does not take into account the leakage of secret key information through side channels. Examples of side channels are execution time [10], computational faults [1] and power consumption [11, 15, 16]. The mentioned systems are vulnerable since the task of monitoring the power consumption or the execution time of cryptographic protocols running on them is relatively easy.

Power Analysis attacks are those based on monitoring and analyzing the power consumption of devices while executing a cryptographic algorithm in order to obtain significant information about the secret key. Kocher *et al.* were the first to present attacks based on *simple* and *differential* power analysis (referred to as SPA and DPA respectively). In SPA, a single power trace of a cryptographic execution is measured and analyzed to identify large features such as iterations of a loop. Specific operations that are typical of a certain cryptosystem like ECC (or exponentiation based cryptosystems) such as point doubling (squaring) or point addition (multiplication) can also be observed. On the other hand, to mount a DPA attack, an attacker collects hundreds of power signal traces, divides them into two groups by using a classification criterion, and calculates the difference between the averaged values of the two groups. Since averaging usually is used in reducing noisy components and strengthening the *differential* signal, DPA is in general more powerful than SPA.

Coron [2] has explained how power analysis attacks can be mounted on ECCs and suggested countermeasures against both types of attacks. Other authors have proposed DPA countermeasures for both general and specific types of elliptic curves [3, 5, 6, 12, 19, 20]. Specifically, the approach followed by the authors in [3, 20] was based each on randomizing the number and the sequence of execution of operations in the scalar multiplication algorithm. This randomization consists of inserting a random decision in the process of building the representation of the scalar k. The algorithms to which this randomization is applied were originally proposed to speed up the EC scalar multiplication. They are based on replacing the binary representation of the *scalar* k by another representation with a fewer number of nonzero symbols by allowing negative symbols to be inserted (*e.g.*, *binary signed-digit* (BSD) representation of an integer). This yields a fewer number of executions of the point addition (or subtraction) in the EC scalar multiplication algorithm. This speedup was possible using the fact that the negative of a point on an elliptic curve is available at no extra computational cost.[1] The algorithms proposed were computing what we referred to as the *canonical* BSD (also known as the *sparse* or NAF) representation of the integer k by scanning the bits of its binary representation starting from the least significant bit. A brief overview of the EC scalar multiplication and the possible power attacks on it is presented in Section 2.

When we consider the DPA countermeasures based on randomizing the BSD representation of the integer k some natural questions arise, those are: What is the average number of BSD representations for an integer k represented in n bits, where the BSD representations are to be of length l signed bits —which we will refer to as *trits* following [8, p.190]—for both cases $l = n$ and $l = n + 1$? For integers represented in n bits, which one has the maximum number of BSD representations since this may affect the choice of the secret scalar

[1]We should note here that the same approach can be appplied in the context of exponentiation-based cryptosystems. But in these systems, using a representation of an integer with negative symbols may be more costly than the traditional *square-and-multiply* method. This is because, for negative symbols of the exponent, a field division—hence, a field inversion—is performed. Therefore, signed-digit representation of the exponent is not used except, possibly, when scanned from left to right as in Algorithm 2.2 where the division is done by a constant.

for cryptosystems that will adopt this countermeasure? In this report, we are interested in answering those questions and the answers are presented in Section 3. Also, in the same section, we present an algorithm that calculates the exact number of those representations for any integer k in $\mathcal{O}(n)$.

In Section 4, we present an algorithm that generates a random BSD representation for an integer k by scanning its bits starting from the most significant bit in $\mathcal{O}(n)$ and another algorithm that generates all BSD representations for an integer k in $\mathcal{O}(3^{\lfloor \frac{n}{2} \rfloor})$ in the worst case. We also demonstrate the effect of increasing n on the number of BSD representations of k.

2 EC Scalar Multiplication and Power Analysis Attacks

In the context of ECCs, the EC scalar multiplication is the core cryptographic operation performed. This operation computes $Q = kP$, *i.e.*, the addition of P to itself k times, where Q and P are two points on a predefined elliptic curve over a finite field and are public, and k is a secret integer. The EC scalar multiplication is conventionally performed using the *double-and-add* algorithms which are also referred to as the *binary algorithms* [8, Section 4.6.3] as depicted in Algorithm 2.1 and Algorithm 2.2. The doubling of a point and the addition of two points on an elliptic curve are operations performed using basic finite field operation as explained in [14].

Algorithm 2.1. Scalar Multiplication using Right-to-Left Double-and-Add

INPUT: k and P
OUTPUT: $Q = kP$

1. $Q \leftarrow \mathcal{O}$
2. for j from 0 to $h-1$ do
   - 2.1 if $(k_j = 1)$ then
     $Q \leftarrow Q + P$
   - 2.2 $P \leftarrow 2P$

Algorithm 2.2. Scalar Multiplication using Left-to-Right Double-and-Add

INPUT: k and P
OUTPUT: $Q = kP$

1. $Q \leftarrow \mathcal{O}$
2. for j from $h-1$ down to 0 do
   - 2.1 $Q \leftarrow 2Q$
   - 2.2 if $(k_j = 1)$ then
     $Q \leftarrow Q + P$

Both Algorithm 2.1 and Algorithm 2.2 scan the bits of the binary representation of the scalar k and accordingly decide whether or not to perform an EC point addition. The difference is in the direction in which they scan these bits. The side channel attacks on the EC scalar multiplication aim to discover the value of k. We assume that the attacker gets hold of the cryptographic token performing the EC scalar multiplication. Coron [2] has explained the possible SPA and DPA attacks that can be mounted on this token. He suggested a reasonable SPA countermeasure at the expense of the execution time of the algorithm. We will focus here on DPA countermeasures.

To mount a DPA attack, the attacker inputs to the token a large number of points and collects the corresponding power traces. As Coron [2] and Hasan [5] have explained, knowing the input point to the algorithm, the attacker starts by guessing the bits of the scalar k starting by the first bit involved in the computation, and accordingly computes the intermediate value for each point he inputs to the algorithm after the bit considered is processed. Based on the representation of the intermediate results, he chooses some partitioning function to partition the power traces he collects and processes them with statistical techniques, such as averaging, to conclude the real value of the bit considered. The attacker then repeats this procedure with every bit of the scalar.

Many of the countermeasures proposed to defeat DPA where based mainly on some form of randomization. They suggested either randomizing (or *blinding*) the point P or the scalar k. For example, one of the countermeasures proposed by Coron [2] suggested randomizing the value of k modulo the order of the point P, this is done as the first step in the scalar multiplication algorithm. Another type of countermeasures, that is of interest to us, is based on modifying the representation of the scalar k throughout the execution of the scalar multiplication algorithm. The representation of k in [3, 20] on which this approach is applied is the BSD representation. The underlying idea is as follows. The right-to-left EC scalar multiplication can be speed up by computing the NAF of k along with performing the corresponding doubling and addition (or subtraction) of points [18]. Random decisions can be inserted in the NAF generating algorithms so that they do not generate the NAF of k but any of the possible BSD representations of k (including the NAF). A different representation for the scalar k is chosen, and hence a different sequence of EC operations is followed, every time the algorithm is executed. We refer the reader to the paper by Oswald and Aigner [20] and the one by Ha and Moon [3] for the details since they are not relevant to the work presented here.

3 Number of Binary Signed Digit Representations

Considering the binary representation of k as one of its BSD representations, different BSD representations for k can be obtained by replacing 01 with $1\bar{1}$ and vice versa and by replacing $0\bar{1}$ with $\bar{1}1$ and vice versa [21, equations 8.4.4- and 8.4.5-]. For example if $k = (11)_{10}$ is represented in $n = 5$ bits, *i.e.*, $k = (01011)_2$, the different BSD representations for k are: 01011, $011\bar{1}1$, $0110\bar{1}$, $1\bar{1}011$, $1\bar{1}1\bar{1}1$, $1\bar{1}10\bar{1}$, $10\bar{1}\bar{1}1$, $10\bar{1}0\bar{1}$. The second representation can be obtained from the first one by replacing the second occurrence of 01 with $1\bar{1}$. The third representation can then be obtained from the second one by replacing the $\bar{1}1$ with $0\bar{1}$, and so forth. Those replacements are done exhaustively till all possible BSD representations for k are obtained.

The binary representation of k, must include at least one 0 that precedes a 1, so that starting from it we can obtain other BSD representations.

We consider in this case positive integers k, *i.e.*, $0 \leq k < 2^n$. In general, the integers x, represented in $l = n$ trits in BSD system, would be in the range $-2^l < x < 2^l$. There are 3^l different combinations for x. Since the BSD representation is balanced [8, Section 4.1], the total number of non-negative combinations is $\frac{3^l+1}{2}$. We will come back to this result later.

3.1 Useful Lemmas

In this subsection, we present a number of lemmas related to the number of BSD representations of an integer k. These lemmas will be used to derive the main results of this article.

Let $\lambda(k,n)$ be the number of BSD representations of k for $0 \leq k < 2^n$ that are $l = n$ trits long. The following lemmas hold.

Lemma 3.1

(i) $\lambda(0,n) = 1$,

(ii) $\lambda(1,n) = n$,

(iii) $\lambda(2^i,n) = n - i$.

Proof.

(i) This is obvious since in two's complement representation, the value 0 is represented with n consecutive zeros. Therefore, there exists no choices for alternative representations.

Let us assume that there is some other BSD representation for the integer $k = 0 = \sum_{i=0}^{n-1} \xi_i 2^i$. Then this representation must contain one or more trits of the value 1 or $\overline{1}$. For example, let us assume that there is a representation where the j^{th} trit is 1, then the summation of the remaining trits with their appropriate weights should be -2^j. This is not possible since there is no appropriate assignment for the ξ_i such that $\sum_{i=0}^{j-1} \xi_i 2^i + \sum_{i=j+1}^{n-1} \xi_i 2^i = -2^j$.

(ii) The possible BSD representations for 1 in n trits are $0^{n-1}1$, $0^{n-2}1\overline{1}$, $0^{n-3}1\overline{1}\overline{1}$, $\ldots$, $1\overline{1}^{n-1}$. Their total number is n.

This is true since, for any $t \in [0,n]$

$$\begin{aligned} 2^t - \sum_{i=0}^{t-1} 2^i &= 2^t - \frac{2^t - 1}{2 - 1} \\ &= 1. \end{aligned}$$

(iii) The possible BSD representations for 2 in n trits are $0^{n-2}10$, $0^{n-3}1\overline{1}0$, $0^{n-4}1\overline{1}\overline{1}0$, $\ldots$, $1\overline{1}^{n-2}0$. Note that these are the same representations for 1 when its binary representation is $n-1$-bit long with an added 0 as the least significant trit. Their total number is $n-1$.
That is,

$$\lambda(2,n) = \lambda(1,n-1) = n - 1.$$

Hence,

$$\lambda(4,n) = \lambda(2,n-1) = \lambda(1,n-2) = n - 2.$$

By induction,

$$\lambda(2^i,n) = n - i.$$

□

Lemma 3.2 *For $2^{n-1} \leq k \leq 2^n - 1$,*

$$\lambda(k, n) = \lambda(k - 2^{n-1}, n - 1).$$

Proof. An integer k in this range would have the binary form $(1\ k_{n-2}\ \ldots\ k_0)_2$ and its value is $2^{n-1} + d$, where $d = (k_{n-2}\ \ldots\ k_0)_2$, and is $(n-1)$ bits long. The BSD representations for d in $n-1$ trits are of the form $(t_{n-2}\ \ldots\ t_0)_{\text{BSD}}$ where $t_{n-2} \in \{0, 1\}$, *i.e.*, $t_{n-2} \neq \overline{1}$ since then the number d would be of negative value.
The BSD representations of k are then of the form $(1\ t_{n-2}\ \ldots\ t_0)_{\text{BSD}}$. The two most significant trits are either 10 or 11. In both cases, no new BSD representations can be generated. Thus, k will have the same BSD representations as for d with an added 1 as the most significant trit. □

Lemma 3.3 *For k even,*

$$\lambda(k, n) = \lambda(\frac{k}{2}, n - 1).$$

Proof. In this case, the integer k is of the form $(k_{n-1}\ \ldots\ k_1\ 0)_2 = 2d$ where $d = (k_{n-1}\ \ldots\ k_1)_2$, and is $(n-1)$ bits long. The BSD representations for d in $n-1$ trits are of the form $(t_{n-1}\ \ldots\ t_1)_{\text{BSD}}$ where $t_1 \in \{\overline{1}, 0, 1\}$.
When d is multiplied by 2, to obtain k, it is shifted left by 1 and a 0 is added to the least significant position. The same is done to each of its BSD representations. In all of them, the least two significant trits will be either $\overline{1}0$, 00 or 10. In all three cases, no new BSD representations can be generated. Thus, k will have the same BSD representations as for d with an added 0 as the least significant trit. □

Lemma 3.4 *For k odd,*

$$\lambda(k, n) = \lambda(k - 1, n) + \lambda(k + 1, n)$$

or

$$\lambda(k, n) = \lambda\left(\left\lfloor \frac{k}{2} \right\rfloor, n - 1\right) + \lambda\left(\left\lfloor \frac{k}{2} \right\rfloor + 1, n - 1\right).$$

Proof. There are two cases to consider.
Case 1: $k \equiv 1 \pmod 4$, that is k ends with 01, $k-1$ ends with 00 and $k+1$ ends with 10. From Lemma 3.3, the BSD representations for $k+1$ in n trits are the same as those for $\frac{k+1}{2}$ with a 0 added as the least significant trit. If this 0 is replaced by a $\overline{1}$, those representations will be possible representations for k. If we think we should start replacing the rightmost $1\overline{1}$ with 01 to generate new representations, we will find out that all representations that end with a 1 will be accounted for by considering those of $k-1$.
Also from Lemma 3.3, the BSD representations for $k-1$ in n trits are the same as those for $\frac{k-1}{4}$ in $n-2$ trits with 00 added as the least significant trits. If the rightmost 0 is replaced with a 1, those representations will be possible representations for k. If we think we should start replacing the rightmost 01 with $1\overline{1}$ to generate new representations, we will find out that all representations that end with a $\overline{1}$ have been accounted for by considering those of $k+1$ as mentioned before.
Case 2: $k \equiv 3 \pmod 4$, that is k ends with 11, $k-1$ ends with 10 and $k+1$ ends with 00. The same argument holds as in case 1. The BSD representations of $k-1$ can be possible representations for k by replacing the least significant 0 with 1. Also the BSD representations for $k+1$ can be possible representations for k by replacing the least significant 0 with $\overline{1}$.

There are no other possible representations for k. This is obvious from the fact that any BSD representation for k should have the rightmost trit either $\overline{1}$ or 1, it can not be 0. Otherwise, $k \equiv 2 \pmod 2$ which is not the case since k is odd. So, if the rightmost 1 or $\overline{1}$ in any representation of k is replaced with 0 then this will be one of the representations of the even number preceding or following k respectively. Hence, using Lemma 3.3, the second form of this lemma is obtained. □

3.2 Number of BSD Representation of length $l = n$

Here we will investigate the total number of BSD representations for all integers in the range $[0, 2^n - 1]$ that are l trits long, where in this case $l = n$. We will call that total number $\sigma(n)$. Table 1 gives an example of $\lambda(k, n)$ and $\sigma(n)$ for small n.

Table 1: $\lambda(k, n)$ and $\sigma(n)$ for small n.

$n=2$		$n=3$		$n=4$		$n=5$		$n=5$ *cont'd*	
k	$\lambda(k,n)$	k	$\lambda(k,n)$	k	$\lambda(k,n)$	k	$\lambda(k,n)$	k	$\lambda(k,n)$
0	1	0	1	0	1	0	1	16	1
1	2	1	3	1	4	1	5	17	4
2	1	2	2	2	3	2	4	18	3
3	1	3	3	3	5	3	7	19	5
		4	1	4	2	4	3	20	2
		5	2	5	5	5	8	21	5
		6	1	6	3	6	5	22	3
		7	1	7	4	7	7	23	4
				8	1	8	2	24	1
				9	3	9	7	25	3
				10	2	10	5	26	2
				11	3	11	8	27	3
				12	1	12	3	28	1
				13	2	13	7	29	2
				14	1	14	4	30	1
				15	1	15	5	31	1
$\sigma(n) = 5$		$\sigma(n) = 14$		$\sigma(n) = 41$				$\sigma(n) = 122$	

We have mentioned at the beginning of this section that this total number, $\sigma(n)$, should be equal to $\frac{3^n+1}{2}$. We will now use the previous lemmas to prove that

$$\sum_{k=0}^{2^n-1} \lambda(k, n) = \sigma(n) = \frac{3^n + 1}{2}.$$

Let

$$\varepsilon(n) = \sum_{k=1}^{2^{n-1}-1} \lambda(k, n). \tag{1}$$

From Lemma 3.1(i) and Lemma 3.2, we have

$$\sigma(n) = \sigma(n-1) + \varepsilon(n) + 1. \tag{2}$$

If we substitute for $\sigma(n)$ recursively in this equation we obtain

$$\sigma(n) = \sigma(1) + \varepsilon(2) + \cdots + \varepsilon(n) + n - 1. \tag{3}$$

From Lemma 3.3 we have

$$\begin{aligned}
\sum_{k=1,\ k \text{ even}}^{2^{n-1}-1} \lambda(k,n) &= \sum_{k=2,\ k \text{ even}}^{2^{n-1}-2} \lambda(\frac{k}{2}, n-1) \\
&= \sum_{i=1}^{2^{n-2}-1} \lambda(i, n-1) \\
&= \varepsilon(n-1).
\end{aligned} \tag{4}$$

From Lemma 3.4 we have

$$\begin{aligned}
\sum_{k=1,\ k \text{ odd}}^{2^{n-1}-1} \lambda(k,n) &= \sum_{k=1,\ k \text{ odd}}^{2^{n-1}-1} \big(\lambda(k-1,n) + \lambda(k+1,n)\big) \\
&= \sum_{i=0,\ i \text{ even}}^{2^{n-1}-2} \lambda(i,n) + \sum_{i=2,\ i \text{ even}}^{2^{n-1}} \lambda(i,n).
\end{aligned}$$

From Lemma 3.1(iii), $\lambda(2^{n-1}, n) = 1$, and also from Lemma 3.1(i) we have

$$\begin{aligned}
\sum_{k=1,\ k \text{ odd}}^{2^{n-1}-1} \lambda(k,n) &= 2 + 2 \sum_{i=2,\ i \text{ even}}^{2^{n-1}-2} \lambda(i,n) \\
&= 2 + 2\,\varepsilon(n-1)
\end{aligned} \tag{5}$$

where the last equality follows from (4). Substituting from equations (4) and (5) into equation (1), we have

$$\begin{aligned}
\varepsilon(n) &= \varepsilon(n-1) + 2 + 2\,\varepsilon(n-1) \\
&= 3\,\varepsilon(n-1) + 2.
\end{aligned} \tag{6}$$

With recursive substitution for $\varepsilon(n)$ and the fact that $\varepsilon(1) = 0$, we obtain

$$\begin{aligned}
\varepsilon(n) &= 3^{n-1}\varepsilon(1) + 2 + 2 \cdot 3 + 2 \cdot 3^2 + \cdots + 2 \cdot 3^{n-2} \\
&= 2 \cdot \frac{3^{n-1} - 1}{3 - 1} \\
&= 3^{n-1} - 1.
\end{aligned} \tag{7}$$

Finally we use 7 and the fact that $\sigma(1) = 2$ to evaluate (3)

$$\begin{aligned}
\sigma(n) &= 2 + (3-1) + \cdots + (3^{n-1} - 1) + n - 1 \\
&= 2 + (n-1) - (n-1) + 3 \cdot \frac{3^{n-1} - 1}{3 - 1} \\
&= \frac{3^n}{2} - \frac{3}{2} + 2 \\
&= \frac{3^n + 1}{2}.
\end{aligned} \tag{8}$$

3.3 Number of BSD Representations of length $l = n + 1$

The algorithms that we are concerned with as DPA countermeasures are based on algorithms that generate the NAF representation of an integer. Since the NAF of an integer may be one trit longer than its binary representation, we are interested in knowing the number of BSD representations of an integer k that are l trits long, where in this case $l = n + 1$.

A part of these representations are the ones discussed previously that were n trits long and their number is $\lambda(k, n)$. For $l = n + 1$, those representations will have a 0 as the most significant trit. If we were to change this 0 to a 1, *i.e.*, add 2^n to the value of k, we should subtract $2^n - k$ in the remaining n trits, that is the 2's complement of k. $2^n - k$ has $\lambda(2^n - k, n)$ BSD representations of length n. The negative of these representations is obtained by interchanging 1 an $\overline{1}$. Thus, if we let $\delta(k, n)$ represent the number of BSD representations of k in $n + 1$ trits for $1 \le k < 2^n$, we have

$$\delta(k, n) = \lambda(k, n) + \lambda(2^n - k, n). \tag{9}$$

The same argument applies to the 2's complement of k

$$\begin{aligned} \delta(2^n - k, n) &= \lambda(2^n - k, n) + \lambda(k, n) \\ &= \delta(k, n). \end{aligned} \tag{10}$$

For $k = 0$, $\delta(0, n) = \lambda(0, n) = 1$. From (10), we conclude that, for k in the defined range, the distribution of $\delta(k, n)$ is symmetric around $k = 2^{n-1}$.

Let $\varsigma(n)$ be the total number of BSD representations of length $n + 1$ for all integers $k \in [0, 2^n - 1]$, we have

$$\begin{aligned} \varsigma(n) &= \sum_{k=0}^{2^n-1} \delta(k, n) = 1 + \sum_{k=1}^{2^n-1} \delta(k, n) \\ &= 1 + \sum_{k=1}^{2^n-1} \big(\lambda(k, n) + \lambda(2^n - k, n)\big) = 1 + 2 \sum_{k=1}^{2^n-1} \lambda(k, n) \\ &= 1 + 2 \left(\frac{3^n + 1}{2} - 1\right) \\ &= 3^n. \end{aligned} \tag{11}$$

Remark 3.1 *We conclude that the average number of BSD representations of an integer, i.e., of possible random forms it can take, is roughly* $\left(\frac{3}{2}\right)^n$

Table 2 gives an example of $\delta(k, n)$ and $\varsigma(n)$ for small n. It is clear from Table 1 and Table 2 and from the definitions of $\lambda(k, n)$ and $\sigma(k, n)$ that, for $0 \le k < 2^{n-1}$,

$$\lambda(k, n) = \sigma(k, n - 1) \tag{12}$$

since in this range, the binary representation of k has a 0 as the leftmost bit. In other words, the algorithm that computes $\lambda(k, n)$ presented in the following section can be used to compute $\sigma(k, n)$ as

$$\sigma(k, n) = \lambda(k, n + 1). \tag{13}$$

Table 2: $\delta(k,n)$ and $\varsigma(n)$ for small n.

$n=2$		$n=3$		$n=4$		$n=5$		$n=5$ *cont'd*	
k	$\delta(k,n)$	k	$\delta(k,n)$	k	$\delta(k,n)$	k	$\delta(k,n)$	k	$\delta(k,n)$
0	1	0	1	0	1	0	1	16	2
1	3	1	4	1	5	1	6	17	9
2	2	2	3	2	4	2	5	18	7
3	3	3	5	3	7	3	9	19	12
		4	2	4	3	4	4	20	5
		5	5	5	8	5	11	21	13
		6	3	6	5	6	7	22	8
		7	4	7	7	7	10	23	11
				8	2	8	3	24	3
				9	7	9	11	25	10
				10	5	10	8	26	7
				11	8	11	13	27	11
				12	3	12	5	28	4
				13	7	13	12	29	9
				14	4	14	7	30	5
				15	5	15	9	31	6
$\varsigma(n)=9$		$\varsigma(n)=27$		$\varsigma(n)=81$				$\varsigma(n)=243$	

3.4 Algorithm to Compute the Number of BSD Representations for an Integer

Here we will present an algorithm that computes $\lambda(k,n)$ for any integer $k \in [0, 2^n - 1]$. This algorithm is based on the lemmas presented in Section 3.1.

Algorithm 3.1. Number of BSD representations of an integer k in $l = n$ trits

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INPUT: $k \in [0, 2^n - 1]$, n

OUTPUT: $C = \lambda(k,n)$

external $\lambda 2(k_e, k_o, w_e, w_o, n)$ /*computed by Algorithm 3.2 that follows*/

1. if $(k = 0)$ then
   $C \leftarrow 1$
2. else if $(k = 1)$ then
   $C \leftarrow n$
3. else if $(k \geq 2^{n-1})$ then
   $C \leftarrow \lambda(k - 2^{n-1}, n-1)$
4. else if (k is even) then
   $C \leftarrow \lambda(\frac{k}{2}, n-1)$
5. else
   5.1 if $(k \equiv 1 \pmod 4)$ then
   $C \leftarrow \lambda 2(\lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1, 1, 1, n-1)$
   5.2 else
   $C \leftarrow \lambda 2(\lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor, 1, 1, n-1)$
6. return C

Algorithm 3.1 uses Lemma 3.1(i) and 3.1(ii) to return the value of $\lambda(k, n)$ directly if the value of k is either 0 or 1. Otherwise, it uses Lemmas 3.2 and 3.3 to trim k recursively from any leading 1's or trailing 0's since they don't add to the number of BSD representations of k as mentioned in the proofs of these Lemmas. Then this algorithm calls Algorithm 3.2 to find the actual number of BSD representations of k which is then an odd integer in the range $[0, 2^{n'-1} - 1]$, for some $n' \leq n$. Hence, Lemma 3.4 is applicable to this k.

Algorithm 3.2. Auxiliary algorithm used by Algorithm 3.1 to compute $\lambda(k, n)$

INPUT: k_e, k_o, w_e, w_o, n
OUTPUT: $c = \lambda 2(k_e, k_o, w_e, w_o, n)$

1. if ($k_o = 1$ AND $k_e = 2$) then
   $c \leftarrow n * w_o + (n-1) * w_e$
2. else
   2.1 if ($k_e \equiv 0 \pmod 4$) then
      2.1.1 if ($k_o \equiv 1 \pmod 4$) then
         $c \leftarrow \lambda 2(\frac{k_e}{2}, \frac{k_e}{2} + 1, w_o + w_e, w_o, n-1)$
      2.1.2 else
         $c \leftarrow \lambda 2(\frac{k_e}{2}, \frac{k_e}{2} - 1, w_o + w_e, w_o, n-1)$
   2.2 else
      2.2.1 if ($k_o \equiv 1 \pmod 4$) then
         $c \leftarrow \lambda 2(\frac{k_e}{2} - 1, \frac{k_e}{2}, w_o, w_o + w_e, n-1)$
      2.2.2 else
         $c \leftarrow \lambda 2(\frac{k_e}{2} + 1, \frac{k_e}{2}, w_o, w_o + w_e, n-1)$
3. return c

Using Lemma 3.4, $\lambda(k, n)$ for k odd constitutes of two other evaluations of the same function λ; one is for an even integer, k_e which is the closest even integer to $k/2$, and the other is for the preceding or the following odd integer, k_o. If we start using Lemmas 3.3 and 3.4 recursively to evaluate λ for k_e and k_o respectively, at each iteration there will be always two terms for the λ function multiplied by a certain weight each, w_e and w_o.

In general, at the i^{th} iteration

$$\lambda(k, n) = w_{e,n-i}\ \lambda(k_{e,n-i}, n-i) + w_{o,n-i}\ \lambda(k_{o,n-i}, n-i).$$

At the beginning, $w_{e,n} = w_{o,n} = 1$. From Lemma 3.3 we have

$$\lambda(k_{e,n-i}, n-i) = \lambda\left(\frac{k_{e,n-i}}{2}, n-i-1\right).$$

From Lemma 3.4 we have

$$\lambda(k_{o,n-i}, n-i) = \lambda\left(\left\lfloor\frac{k_{o,n-i}}{2}\right\rfloor, n-i-1\right) + \lambda\left(\left\lfloor\frac{k_{o,n-i}}{2}\right\rfloor + 1, n-1\right).$$

There are two possible cases for k_e in each iteration and two corresponding subcases for k_o.

Case 1: $k_{e,n-i} \equiv 0 \pmod 4 \quad \Rightarrow \quad \frac{k_{e,n-i}}{2}$ is even

Case 1.1: $k_{o,n-i} \equiv 1 \pmod 4$, *i.e.*, $k_{0,n-i} = k_{e,n-i} + 1 \Rightarrow \left\lfloor \frac{k_{o,n-i}}{2} \right\rfloor$ is even
Hence,
$k_{e,n-i-1} = \left\lfloor \frac{k_{o,n-i}}{2} \right\rfloor = \frac{k_{o,n-i}-1}{2} = \frac{k_{e,n-i}}{2}$
$k_{o,n-i-1} = \left\lfloor \frac{k_{o,n-i}}{2} \right\rfloor + 1 = \frac{k_{o,n-i}+1}{2} = \frac{k_{e,n-i}}{2} + 1$

Case 1.2: $k_{o,n-i} \equiv 3 \pmod 4$, *i.e.*, $k_{0,n-i} = k_{e,n-i} + 1 \Rightarrow \left\lfloor \frac{k_{o,n-i}}{2} \right\rfloor$ is odd
Hence,
$k_{e,n-i-1} = \left\lfloor \frac{k_{o,n-i}}{2} \right\rfloor + 1 = \frac{k_{o,n-i}+1}{2} = \frac{k_{e,n-i}}{2} + 1$
$k_{o,n-i-1} = \left\lfloor \frac{k_{o,n-i}}{2} \right\rfloor = \frac{k_{o,n-i}-1}{2} = \frac{k_{e,n-i}}{2}$

In case 1, the weights are updated as follows
$w_{e,n-i-1} = w_{o,n-i} + w_{e,n-i}$
$w_{o,n-i-1} = w_{o,n-i}$

Case 2: $k_{e,n-i} \equiv 2 \pmod 4 \quad \Rightarrow \quad \frac{k_{e,n-i}}{2}$ is odd

Case 2.1: $k_{o,n-i} \equiv 1 \pmod 4$, *i.e.*, $k_{0,n-i} = k_{e,n-i} - 1$
Hence,
$k_{e,n-i-1} = \left\lfloor \frac{k_{o,n-i}}{2} \right\rfloor = \frac{k_{o,n-i}-1}{2} = \frac{k_{e,n-i}}{2} - 1$
$k_{o,n-i-1} = \left\lfloor \frac{k_{o,n-i}}{2} \right\rfloor + 1 = \frac{k_{o,n-i}+1}{2} = \frac{k_{e,n-i}}{2}$

Case 2.2: $k_{o,n-i} \equiv 3 \pmod 4$, *i.e.*, $k_{0,n-i} = k_{e,n-i} + 1$
Hence,
$k_{e,n-i-1} = \left\lfloor \frac{k_{o,n-i}}{2} \right\rfloor + 1 = \frac{k_{o,n-i}+1}{2} = \frac{k_{e,n-i}}{2} + 1$
$k_{o,n-i-1} = \left\lfloor \frac{k_{o,n-i}}{2} \right\rfloor = \frac{k_{o,n-i}-1}{2} = \frac{k_{e,n-i}}{2}$

In case 2, the weights are updated as follows
$w_{e,n-i-1} = w_{o,n-i}$
$w_{o,n-i-1} = w_{o,n-i} + w_{e,n-i}$

Following are some examples that illustrate the iterations of Algorithm 3.2 for some chosen values of k. In these examples we have used the notation (k) that means $\lambda(k, n)$. We have also used a tree representation where, for a parent node k, the number of BSD representations is equal to the sum of the BSD representations of its child nodes. Each level of the tree represents the updating of weights at each iteration. The examples are chosen such that k_e and k_o that are passed as arguments to Algorithm 3.2, *e.g.*, 12 and 13 in figure 1, correspond to the four different cases discussed.

For the complexity of Algorithm 3.1—including its usage of Algorithm 3.2—it is clear that it runs in $O(n)$ time and occupies $O(n)$ bits in memory. This time complexity results from the fact that both algorithms deal with the integer k one bit at a time. For the space complexity of Algorithm 3.1, even in its recursive form, the new values that it generates for k and n can replace the old values in memory. The same argument is valid for Algorithm 3.2 regarding the new values of k_e, k_o, w_e, w_o and n.

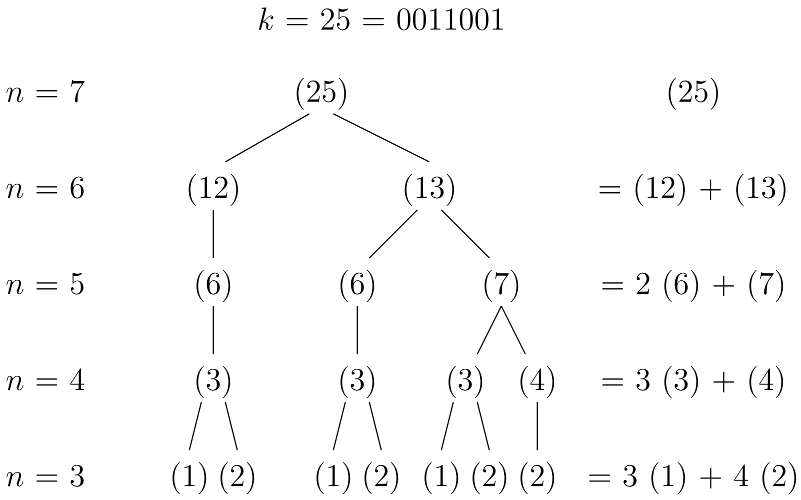

Figure 1: $\lambda(25, 7)$, corresponds to case 1.1.

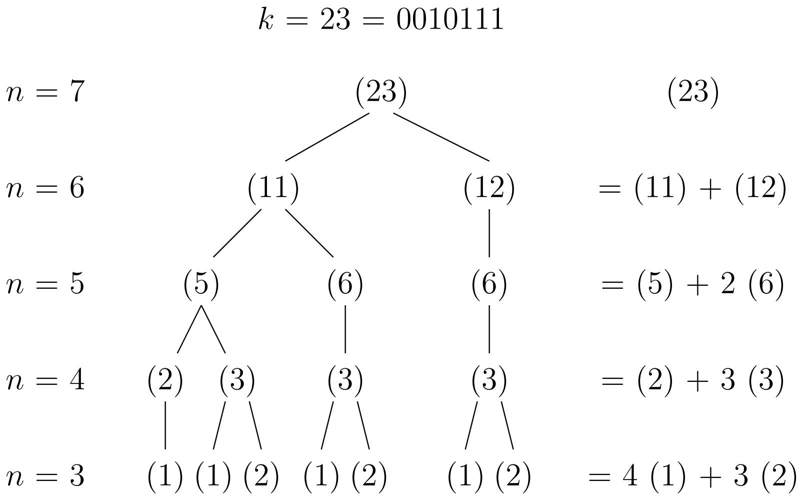

Figure 2: $\lambda(23, 7)$, corresponds to case 1.2.

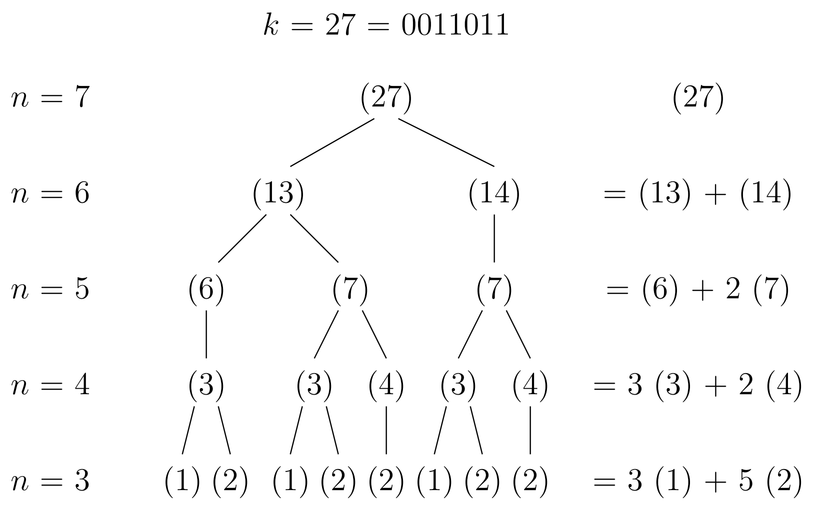

Figure 3: $\lambda(27, 7)$, corresponds to case 2.1.

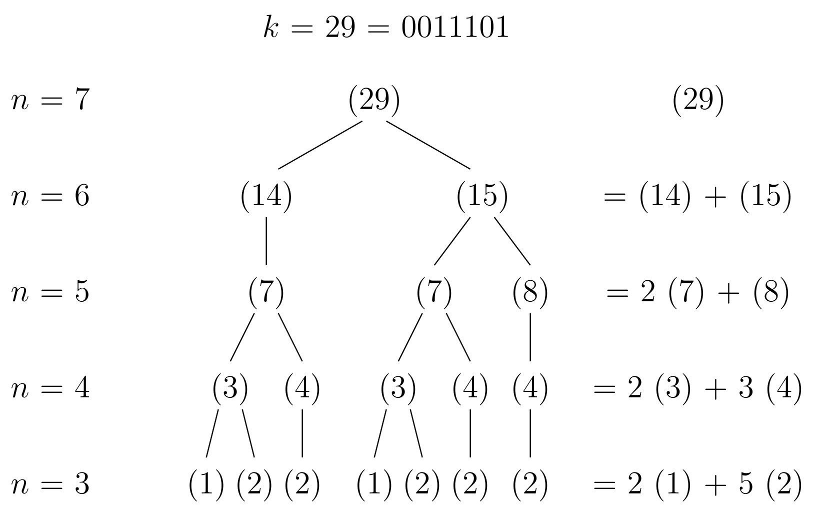

Figure 4: $\lambda(29, 7)$, corresponds to case 2.2.

3.5 Integer with Maximum Number of BSD Representations

From the view point of DPA countermeasure, while choosing a secret key k, it is desirable to know which integer $k \in [0, 2^n - 1]$, for a certain number of trits $l = n$ or $l = n + 1$, has the maximum number of BSD representations. We note from (12) that this integer with the largest $\delta(k, n)$ is the same one with the largest $\lambda(k, n + 1)$. Also from the symmetry of $\delta(k, n)$ around 2^{n-1}, we note that there are two values of k for which $\delta(k, n)$ is the maximum value. We will denote them as $k_{max_1,n}$ and $k_{max_2,n}$. $k_{max_1,n}$ and $k_{max_2,n}$ are equidistant from 2^{n-1}. Therefore, we will only derive a formula for $k_{max_1,n}$.

For example, as shown in Table 1 and Table 2, for $n = 2$, $k_{max_1,n} = 1$ and $k_{max_2,n} = 3$. For $n = 3$, $k_{max_1,n} = 3$ and $k_{max_2,n} = 5$.

From Lemma 3.3 and Lemma 3.4, $k_{max_1,n}$ and $k_{max_2,n}$ must be odd integers. There are two cases:

Case 1: $k_{max_1,n} \equiv 1 \pmod 4$

In this case, $\sigma(k_{max_1,n} + 1, n) > \sigma(k_{max_1,n} - 1, n)$, since $k_{max_1,n} - 1$ has two zeros as the least significant trits while $k_{max_1,n} + 1$ has only one zero. Those rightmost zeros are trimmed by Algorithm 3.1.

Thus, from Lemma 3.4, we have

$$\begin{aligned} k_{max_1,n+1} &= k_{max_1,n} + k_{max_1,n} + 1 \\ &= 2\ k_{max_1,n} + 1 \\ &\equiv 3 \pmod 4. \end{aligned} \tag{14}$$

Case 2: $k_{max_1,n} \equiv 3 \pmod 4$

In this case, $\sigma(k_{max_1,n} - 1, n) > \sigma(k_{max_1,n} + 1, n)$. Thus, we have

$$\begin{aligned} k_{max_1,n+1} &= k_{max_1,n} + k_{max_1,n} - 1 \\ &= 2\ k_{max_1,n} - 1 \\ &\equiv 1 \pmod 4. \end{aligned} \tag{15}$$

With every increment of n, cases 1 and 2 alternate. Case 1 occurs when n is even and Case 2 occurs when n is odd.

For n even ($n - 1$ odd), we can substitute from (14) into (15) to obtain

$$\begin{aligned} k_{max_1,n} &= 2\ k_{max_1,n-1} - 1 \\ &= 2\ (2\ k_{max_1,n-2} + 1) - 1 \\ &= 4\ k_{max_1,n-2} + 1 \end{aligned}$$

Using recursive substitution,

$$\begin{aligned} k_{max_1,n} &= 4 \cdot 4 \cdot 4 \cdots k_{max_1,2} + 1 + 4 + 16 + \cdots \\ &= (2^2)^{\frac{n-2}{2}} \cdot 1 + \frac{(2^2)^{\frac{n-2}{2}} - 1}{2^2 - 1} \\ &= \frac{1}{3}(2^n - 1). \end{aligned} \tag{16}$$

For n odd ($n-1$ even), we can follow the same derivation procedure to obtain

$$k_{max_1,n} = \frac{1}{3}(2^n + 1). \tag{17}$$

Since, as was mentioned before, $k_{max_1,n}$ and $k_{max_2,n}$ are equidistant from 2^{n-1}, we can obtain $k_{max_2,n}$ for n even as follows

$$\begin{aligned} k_{max_2,n} &= 2^{n-1} + 2^{n-1} - k_{max_1,n} \\ &= 2^n - \frac{1}{3}(2^n - 1) \\ &= \frac{1}{3}(2^{n+1} + 1) \\ &= k_{max_1,n+1} \end{aligned} \tag{18}$$

where the last equality follows from (17).

Similarly, for n odd, we have

$$\begin{aligned} k_{max_2,n} &= \frac{1}{3}(2^{n+1} - 1) \\ &= k_{max_1,n+1}. \end{aligned} \tag{19}$$

Finally we should notice that, for n even, $k_{max_1,n}$ is of the form $(\langle 0\ 1\rangle^{\frac{n}{2}})_2$ and, for n odd, $k_{max_1,n}$ is of the form $(\langle 0\ 1\rangle^{\frac{n-1}{2}} 1)_2$.

Knowing the integer with maximum number of BSD representations can help the cryptographer choose a secret key as follows. He can run Algorithm 3.1 with that integer as input to know the maximum number of representations. Then, he can pick an integer at random and run the same algorithm again to compare the number of BSD representations of that integer with the maximum one and accept that integer as the secret key if the number of its representations is within a predefined range of the maximum number. In general, from Lemma 3.4, odd integers have a number of BSD representations of order of twice that of their neighboring even integers. We have also observed that integers having a good random distribution of their bits are more probable to have larger number of BSD representations.

4 Left-to-Right BSD Randomization and Generation Algorithms

The algorithms mentioned in Section 2 that are used to generate a random BSD representation of k are modified versions of NAF generation algorithms. Those latter ones scan the bits of the integer k from right-to-left and hence, perform the EC scalar multiplication from right to left.

Sometimes, it is advantageous to perform the EC scalar multiplication from left to right, especially when a mixed coordinate (projective and affine) system is used for saving computational cost in scalar multiplication [4, 13]. Joye and Yen [7] have introduced in their paper an algorithm that generates the NAF of an integer and another algorithm that generates a minimal weight representation of it. That is the output of the second algorithm has the same weight as the NAF of the number but is not necessarily *scarce*. The difficulty of their first algorithm was that it can decide what the output will be for a scanned part of the integer only after encountering two consecutive equal bits. Hence, it could not be implemented in a look-up table method as the original Reitweisner's algorithm [21]. Their second algorithm overcame this difficulty.

In this section, we will present an algorithm that generates a random $n+1$-bit BSD representation of k while scanning it from left to right. Also we will present another algorithm that generates all BSD representations of k in $n+1$ bits.

4.1 Left-to-Right Randomization Algorithm

We start with an example. Let $k = 10001001 = 2^7 + 2^4 + 2^0$. As in the proof of Lemma 3.1(ii), the least significant 1 can have up to n BSD representations in n trits. If we consider that k is the sum of these three integers as follows

$$\begin{aligned} k &= 10000000 \\ &+ 00001000 \\ &+ 00000001 \end{aligned}$$

and we consider the different BSD representations for each integer separately and then adding those different representations together to get those of k.

The first integer has no other representation according to Lemma 3.1(iii). From the same lemma, the second integer has the following additional representations $0001\bar{1}000$, $001\bar{1}\bar{1}000$, $01\bar{1}\bar{1}\bar{1}000$ and $1\bar{1}\bar{1}\bar{1}\bar{1}000$. If this last one were to be added to the first integer, then k would need more than 8 trits in this example to represent it, so we will not take it into consideration. Finally the third integer has this set of additional representations $0000001\bar{1}$, $000001\bar{1}\bar{1}$, $00001\bar{1}\bar{1}\bar{1}\ldots$, $01\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}\bar{1}$. We will notice that we only need to take into account the first three representations. This is because all the representations starting from the third one when added to the first two integers will yield a 0 in trit position 3, $\bar{1}$s in the lower positions and a trit pattern in the upper positions that has been accounted for in the different representations of the second integer.

Also, we can consider that this representation of k

$$\begin{aligned} k &= 10000000 \\ &+ 001\bar{1}\bar{1}000 \\ &+ 00001\bar{1}\bar{1}\bar{1} \\ \hline &= 101\bar{1}0\bar{1}\bar{1}\bar{1} \end{aligned}$$

could be obtained from this one

$$\begin{aligned} k &= 10000000 \\ &+ 001\bar{1}\bar{1}000 \\ &+ 000001\bar{1}\bar{1} \\ \hline &= 101\bar{1}\bar{1}1\bar{1}\bar{1} \end{aligned}$$

after changing the $\bar{1}1$ at trit positions 3 and 2 to $0\bar{1}$.

Thus, the underlying idea of the algorithm is that the binary representation of k is subdivided into groups of bits—of different lengths—such that each group is formed by a number of consecutive 0s ending with a single 1. For each of these groups a random BSD representation is chosen as in the proof of Lemma 3.1(ii). Whenever the choice yields a $\bar{1}$ at the end of a group—which happens when any representation for that group other than the binary one is chosen—and a 1 at the beginning of the next group—which happens when the representation chose for the next group is the one that has no 0s—another random decision is taken so as whether to leave those two trits (*i.e.*,, $\bar{1}1$) as they are or to change them to $0\bar{1}$.

The choice of a random representation for a certain group is done by counting the number of 0s in it, say z, and choosing a random integer $t \in [0, z]$ which will be the number of 0s to be written to the output. If t is equal to z, the last trit to be written to the output for this group is 1, and it is actually written. Otherwise, after writing the t 0s, a 1 and only $z - t - 1$ $\bar{1}$s are written, that is the last $\bar{1}$ is not written, but saved as the value of a variable labeled as *last*. We then do the same for the next group. If for the next group $t = 0$, we take a random decision whether to write $\bar{1}1$ to the output or $0\bar{1}$ at the boundary of the two groups. This leads to the following algorithm.

Algorithm 4.1. Left-to-Right Randomization of an integer's BSD representation

---

INPUT: $k = (k_{n-1}\ \ldots\ k_0)_2$
OUTPUT: $k' = (k'_n\ \ldots\ k'_0)_{\mathrm{BSD}}$, a random BSD representation of k

1. Set $k_n \leftarrow 0;\ i \leftarrow n+1;\ last \leftarrow 1$
2. for j from n down to 0 do
   if $(k_j = 1)$ then
   - 2.1 $t \leftarrow_R [0, i-j-1]$
   - 2.2 if $(t = 0$ AND $last \neq 1)$ then
     - 2.2.1 $c \leftarrow_R \{0, 1\}$
     - 2.2.2 if $(c = 0)$ then
       $k'_i \leftarrow \bar{1}$
       $i \leftarrow i-1;\ k'_i \leftarrow 1$
     - 2.2.3 else
       $k'_i \leftarrow 0$
   - 2.3 else
     - 2.3.1 if $(last \neq 1)$ then
       $k'_i \leftarrow \bar{1}$
     - 2.3.2 while $(t > 0)$ do
       $i \leftarrow i-1;\ k'_i \leftarrow 0;\ t \leftarrow t-1$
     - 2.3.3 $i \leftarrow i-1;\ k'_i \leftarrow 1$
   - 2.4 if $(i = j)$ then
     $last \leftarrow 1$
   - 2.5 else
     - 2.5.1 while $(i > j+1)$ do
       $i \leftarrow i-1;\ k'_i \leftarrow \bar{1}$
     - 2.5.2 $i \leftarrow i-1;\ last \leftarrow \bar{1}$
3. if $(last \neq 1)$ then
   $k'_i \leftarrow \bar{1}$
4. while $(i > 0)$ do
   $i \leftarrow i-1;\ k'_i \leftarrow 0$

The algorithm runs in $O(n)$ time. The bits of k' can be used as they are generated in left-to-right scalar multiplication algorithms in EC cryptosystems or in left-to-right exponentiation algorithms in RSA or ElGamal cryptosystems. This means that there is no need to store k'.

4.2 Left-to-Right Generation Algorithm

The algorithm presented here is a modified version of Algorithm 4.1 that recursively and extensively substitutes every group of 0s ending with a 1 with one of its possible forms. It also takes into consideration the alternative substitutions at the group boundary when the representation of a group ends with a $\overline{1}$ and that of the next group starts with 1. This algorithm can be used as a straight-forward check that Algorithm 4.1 is capable of generating any possible $n+1$-bit BSD representation of an integer k in the range $[1, 2^n - 1]$, *i.e.*, there is no BSD representation of k that cannot be generated by that algorithm. It was tested on small values of n.

Algorithm 4.2. Left-to-Right Generation of all BSD representations of k

---

INPUT: $k = (k_{n-1} \, \ldots \, k_0)_2$
OUTPUT: all possible strings $k' = (k'_n \, \ldots \, k'_0)_{\mathrm{BSD}}$

1. Subdivide k from left to right into groups of consecutive 0s each ending with a single 1. Store the length of each group in a look-up table G. Let g be the index of the table.
2. Set $g \leftarrow 0$; $i \leftarrow n+1$;
   $last \leftarrow 1$; $j \leftarrow i - G[g]$; $k' \leftarrow \langle\rangle$
3. for $t = 0$ to $i - j - 1$ do
   ChooseForm$(k, g, t, i, j, last, k')$

Algorithm 4.3. ChooseForm$(k, g, t, i, j, last, k')$, a recursive procedure employed by Algorithm 4.2

INPUT: k, g, t, i, j, $last$, k'
OUTPUT: returns k', a string of trits, as a possible BSD representation of k.

1. if $(t > 0)$ OR $(last = 1)$ then //this step is equivalent to step 2.3 in Algorithm 4.2
   1.1 if $(last \neq 1)$ then
   $k' \leftarrow k'|\overline{1}$ //concatenate k' with $\overline{1}$
   1.2 while $(t > 0)$ do
   $i \leftarrow i - 1$; $k' \leftarrow k'|0$; $t \leftarrow t - 1$
   1.3 $i \leftarrow i - 1$; $k' \leftarrow k'|1$
2. if $(i = j)$ then
   $last \leftarrow 1$
3. else
   3.1 while $(i > j + 1)$ do
   $i \leftarrow i - 1$; $k \leftarrow k'|\overline{1}$
   3.2 $i \leftarrow i - 1$; $last \leftarrow \overline{1}$
4. if $(j = 0)$ then
   4.1 if $(last \neq 1)$ then
   $k' \leftarrow k'|\overline{1}$

4.2 return k'

5. $g \leftarrow g+1;\; j \leftarrow j - G[g]$

6. if $(j = 0)$ AND (k is even) then

   6.1 if $(i > 0)$ then

   6.1.1 if $(last \neq 1)$ then
   $k' \leftarrow k'|\bar{1}$

   6.1.2 while $(i > 0)$ do
   $i \leftarrow i - 1;\; k' \leftarrow k'|0$

   6.2 return k'

7. $t = 0$

8. if $(last \neq 1)$ then
   ChooseForm$(k, g, t, i-1, j, last, k'|\bar{1}1)$
   ChooseForm$(k, g, t, i, j, last, k'|0)$

9. else
   ChooseForm$(k, g, t, i, j, last, k')$

10. for $t = 1$ to $i - j - 1$ do
    ChooseForm$(k, g, t, i, j, last, k')$

To better explain how this algorithm works we present in Figure 5 the tree explored by the algorithm for $k = 21$ and $n = 5$. This tree is explored by the algorithm in a *depth-first* fashion. That is, the recursive function *ChooseForm* is first called from Algorithm 4.2 at node a in the figure. Then this function calls itself at node b and then at node c where it returns with the first BSD representation for $k = 21$ which is $1\bar{1}1\bar{1}1\bar{1}$. With the flow control at node b the function calls itself at node d where the second BSD representation is generated and so forth.

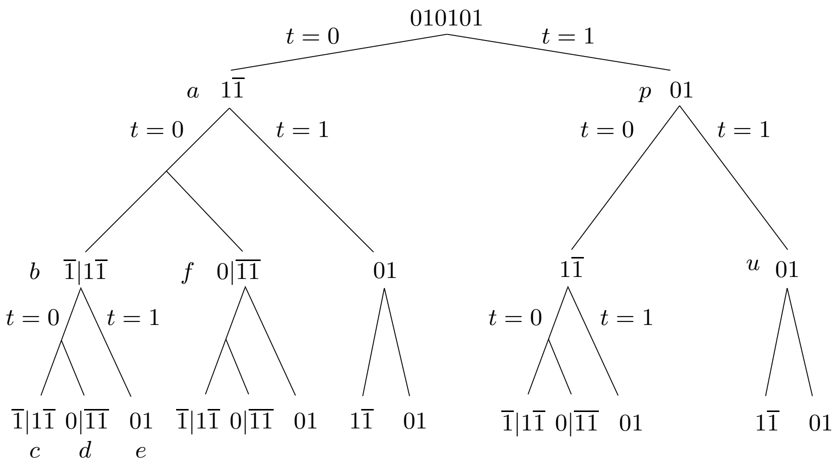

Figure 5: The tree explored by Algoritm 4.2 for $k = 21$ and $n = 5$.

The algorithm works as follows. The integer k is first subdivided into groups of bits where each group consists of consecutive 0s and a single 1 at the end as for Algorithm 4.1. We should mention that a 0 is prepended to the binary representation of k, so that the BSD representations, k' of k are $n+1$-trit long. Starting from the leftmost group, a particular BSD representation for that group is chosen starting with the one that contains no 0s (*i.e.*, t, the number of 0s to be written to the output for that group, is equal to 0). The representation is formed inside the function *ChooseForm* which takes t as one of its arguments. In turn, this function goes through the possible values of t for the following group and, for each one, calls itself to form the corresponding BSD representation of that group. When t is equal to 0, the two possible alternatives at the group boundary are considered as was explained in Section 4.1. For example, in Figure 5, the last trit in the group at node a may remain $\bar{1}$or change to 0 depending on the random decision taken at the group boundary when $t = 0$ for the next group. This is why this last trit is written at nodes b and f before the symbol '|' which designates the boundary between the groups.

The *worst-case* complexity analysis of Algorithm 4.2 is presented in the following. The worst case occurs for the integer with the maximum number of BSD representations in the range $[1, 2^n - 1]$. There are actually two integers with this property for any given n, which we referred to as $k_{max_1,n}$ and $k_{max_2,n}$ in Section 3.5. We mentioned that, for n even, $k_{max_1,n}$ is of the form $(\langle 0\ 1\rangle^{\frac{n}{2}})_2$. For example, $k_{max_1,6} = 21 = (010101)_2$ (see Figure 6). We also mentioned that, for any n, $k_{max_2,n} = k_{max_1,n+1}$. For example, $k_{max_1,5} = 11 = (01011)_2$ and $k_{max_2,5} = 21 = (10101)_2$ (see Figure 6). Therefore, our analysis is conducted on those integers k of the binary form $(\langle 0\ 1\rangle^{\frac{n}{2}})_2$ for n even and $(1\langle 0\ 1\rangle^{\frac{n-1}{2}})_2$ for n odd. In the following discussion, we will drop the subscript n from $k_{max_1,n}$ and $k_{max_2,n}$ for simplicity, since it will be obvious from the context.

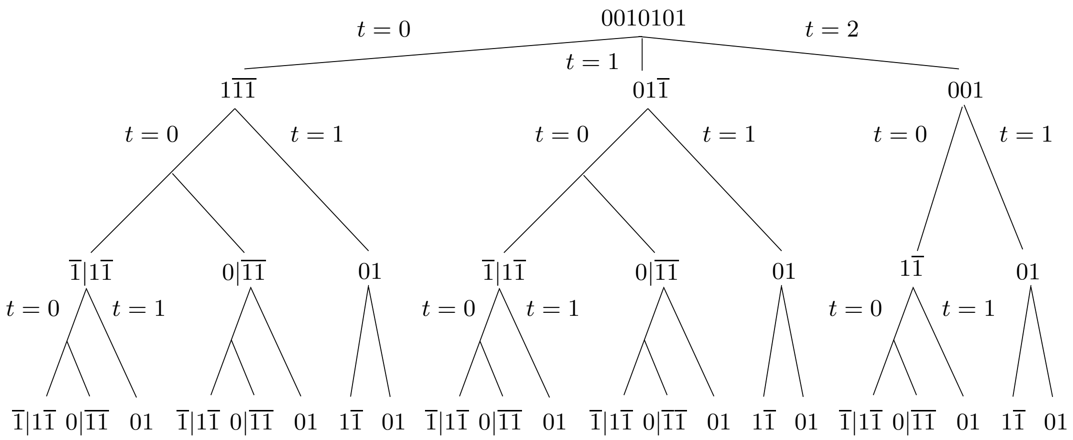

Figure 6: The tree explored by Algoritm 4.2 for $k = 21$ and $n = 6$.

For n even, we have the following.

$n = 2$: $k_{max_1} = k_{max_2} = 1 = (01)_2$
$\sigma(1, 2) = \lambda(1, 3) = 3$
The tree explored here is the same as the one having as root the node b in Figure 5. The difference is that in this case t can take the values 0, 1 and 2 with only one representation for $t = 0$.

$n = 4$: $k_{max_1} = 5 = (0101)_2$
$\sigma(5, 4) = \lambda(5, 5) = 8 = 3 \cdot 3 - 1$
The tree explored for this integer is the same as the one having as root a in Figure 5.

$n = 6$: $k_{max_1} = 21 = (010101)_2$

$\sigma(21,6) = \lambda(21,7) = 21 = 3 \cdot 3 \cdot 3 - (3.1 + 3)$ The tree explored for this integer is illustrated in Figure 6.

Let $m = \frac{n}{2}$. By induction we can deduce the following

$$\begin{aligned}\lambda(k_{max_1}, n+1) = 3^m - (m-1)3^{m-2} + \left(\sum_{i_1=1}^{m-3} i_1\right) 3^{m-4} \\ - \left(\sum_{i_1=1}^{m-5} \sum_{i_2=1}^{i_1} i_2\right) 3^{m-6} + \left(\sum_{i_1=1}^{m-7} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} i_3\right) 3^{m-8} - \cdots\end{aligned} \tag{20}$$

For n odd, we have the following.

$n = 1$: $k_{max_1} = k_{max_2} = 1 = (1)_2$
$\sigma(1,1) = \lambda(1,2) = 2$
The tree explored here is simply the same as the one having as root the node u in Figure 5.

$n = 3$: $k_{max_2} = 5 = (101)_2$
$\sigma(5,3) = \lambda(5,4) = 5 = 2 \cdot 3 - 1$
The tree explored for this integer is the same as the one having as root the node p in Figure 5.

$n = 5$: $k_{max_2} = 21 = (10101)_2$
$\sigma(21,5) = \lambda(21,6) = 13 = 2 \cdot 3 \cdot 3 - (3 + 2 \cdot 1)$
The tree for this integer is the one illustrated in Figure 5.

Let $m = \frac{n-1}{2}$. By induction we can deduce the following

$$\begin{aligned}\lambda(k_{max_2}, n+1) = 2 \cdot 3^m - \left[3^{m-1} + 2(m-1)3^{m-2}\right] \\ + \left[(m-2)3^{m-3} + 2\left(\sum_{i_1=1}^{m-3} i_1\right) 3^{m-4}\right] \\ - \left[\left(\sum_{i_2=1}^{m-4} i_2\right) 3^{m-5} + 2\left(\sum_{i_1=1}^{m-5} \sum_{i_2=1}^{i_1} i_2\right) 3^{m-6}\right] \\ + \left[\left(\sum_{i_2=1}^{m-6} \sum_{i_3=1}^{i_2} i_3\right) 3^{m-7} + 2\left(\sum_{i_1=1}^{m-7} \sum_{i_2=1}^{i_1} \sum_{i_3=1}^{i_2} i_3\right) 3^{m-8}\right] \\ - \cdots\end{aligned} \tag{21}$$

From this discussion, we state the following theorem.

Theorem 4.1 *For any n, the number of BSD representations generated by Algorithm 4.2 is, in the worst case, $\mathcal{O}(3^{\lfloor \frac{n}{2} \rfloor})$.*

4.3 Effect of Increasing n on the Number of BSD Representations of an Integer k

In an attempt to improve DPA countermeasure, one may increase the length of the binary representation of the integer k. Below we show how the number of BSD representations of k increases if we lengthen its binary representation by adding 0s at the most significant end.

If we compare Figure 5 with Figure 6, we see that for the same integer $k = 21$, increasing n from 5 to 6 had the effect of increasing the number of branches emerging from the root by one. The added branch has the same tree structure as the leftmost branch a in Figure 5. This is because the number of BSD representations of the first group—recall how the integer is subdivided into groups—has increased by one. Since all representations of a group, except for the original binary representation, end with a $\overline{1}$, the added representation would generate two alternatives when $t = 0$ for the next group. If we increase n to 7, another subtree like the one having as root a will be added to the tree. The same subtree is repeated with every 0 prepended to the binary representation of k. It is easy to verify that this is true for any integer k.

As was mentioned before, the subtree having as root the node a is the tree explored for $k = 5$ and $n = 4$. In general, the subtree that is repeated is the one formed for the integer with the binary representation having the same groups as k except for the leftmost group. That is

$$\Delta(k) = \lambda(k, n+1) - \lambda(k, n) = \lambda(k - 2^{\lfloor \log_2 k \rfloor}, \lfloor \log_2 k \rfloor + 1) \tag{22}$$

where $\Delta(k)$ is the number of leaves in the repeated subtree.

We notice that $\Delta(k)$ does not depend on n and hence, based on (22), we have the following theorem.

Theorem 4.2 *The number of BSD representations of an integer increases linearly with the number of bits in which it is represented in its binary form.*

5 Conclusion

In this report, we have presented a number of cryptographically important issues related to the binary-signed digit (BSD) representation of integers, such as the average number of BSD representations of an integer k in the range $[0, 2^n - 1]$, the integer in this range that has maximum number of BSD representations. Our results provide a mechanism for choosing integers (*i.e.*, cryptographic keys) with larger space of BSD representations and hence, with better resistance to DPA attacks. We have presented an algorithm that calculates for any integer k represented in n bits the exact number of BSD representations with the same length in $\mathcal{O}(n)$. We have also presented an algorithm that generates a random BSD representation of an integer by scanning its bits starting from the most significant end which can be used where the EC scalar multiplication is to be performed from left to right to reduce computational cost using a mixed coordinate system. This algorithm runs in $\mathcal{O}(n)$. In addition, we presented an algorithm that can generate all BSD representations of an integer, which runs in $\mathcal{O}(3^{\lfloor \frac{n}{2} \rfloor})$ in the worst case, *i.e.*, in the case of the integer with maximum number of BSD representations in the range $[0, 2^n - 1]$. We have also proved that the number of BSD representations of an integer increases linearly with the number of bits in which it is represented.

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