Near-complementary Sequences With Low PMEPR for Peak Power Control in Multicarrier Communications

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Abstract

New families of near-complementary sequences are presented for peak power control in multicarrier communications. A framework for near-complementary sequences is given by the explicit Boolean expression and the equivalent array structure. The framework transforms the seed pairs to near-complementary sequences by the aid of Golay complementary sequences. As the first example, a new sequence family of length 2^m and peak-to-mean envelope power ratio (PMEPR) ≤ 4 is presented, where the family produces more distinct sequences than any other known near complementary sequences of the same lengths and PMEPR bound. An efficient generation algorithm for permutations is developed for the distinct sequences. In addition, new families of near-complementary sequences of various lengths and PMEPR < 4 are also presented, where the sequences are constructed by the framework employing the seeds of shortened or extended Golay complementary pairs. The families present in a constructive way a large number of sequences of PMEPR < 4 for the lengths (< 100) of $24, 28, 30, 34, 36, 48, 56, 60, 62, 66, 68, 72$, and 96 where no Golay pairs have been reported. The sequence families can find the potential applications for peak power control requiring codewords or sequences of various lengths as well as low PMEPRs.

Index Terms

Aperiodic autocorrelation, Boolean functions, Golay complementary sequences, Multicarrier code-division multiple access (MC-CDMA), Multicarrier communications, Orthogonal codes, Orthogonal frequency-division multiplexing (OFDM), Peak-to-average power ratio (PAPR), Peak-to-mean envelope power ratio (PMEPR), Reed-Muller codes, Sequences.

I. INTRODUCTION

Multicarrier communications have recently attracted much attention in wireless applications. The orthogonal frequency division multiplexing (OFDM) has been employed as a multiplexing and a multiple access technique in several wireless communication standards such as IEEE802.11 Wireless LAN [1], IEEE802.16 Mobile WiMAX [2], and 3GPP-LTE [3]. Also, the multicarrier code-division multiple access (MC-CDMA) is of interest as a future wireless multiple access scheme. Their popularity is mainly due to the robustness to multipath fading channels and the efficient hardware implementation employing fast Fourier transform (FFT) techniques. However, multicarrier communications have the major drawback of the high peak-to-average power ratio (PAPR) of transmitted signals, which may nullify all the potential benefits [16].

A constructive way for peak power control in multicarrier communications is to use complementary sequences [9] for subcarriers such that the sequences provide low peak-to-mean envelope power ratio (PMEPR) of at most 2 for transmitted signals [17], where the PAPR of the signals is bounded by the PMEPR [10]. A milestone of the theoretical research on the *Golay complementary sequences* has been set by Davis and Jedwab [6], where they showed the sequences can be constructed by a coset of the first order Reed-Muller codes. Specifically, they reported that 2^h-ary Golay complementary sequences of length 2^m can be obtained by the second order coset of the generalized first order Reed-Muller codes $\mathrm{RM}_{2^h}(1,m)$. Paterson [16] then presented a unified theory for *Golay complementary sets* of size 2^{k+1}, where each sequence in the set has the PMEPR of at most 2^{k+1}. Recently, Fiedler, Jedwab, and Parker [8] identified a framework of constructions of all known Golay complementary sequences of length 2^m by a matrix structure and an explicit Boolean expression.

As an alternative to Golay complementary sequences, numerous approaches have been made for *near-complementary* sequences, where the PMEPR is bounded by a finite value larger than 2. In [13], Parker and Tellambura presented a primitive framework for construction of near-complementary sequences, where they applied the generalized Rudin-Shapiro construction [18][21] to a pair of starting seed sequences with PMEPR $\leq v$ for constructing longer near-complementary sequences with PMEPR $\leq v$. Based on the work, Schmidt [19] gave an explicit Boolean expression for a framework of near-complementary sequences of length 2^m. In general, near-complementary sequences are expected to produce more sequences than the

Golay complementary sequences at the cost of the increase of the PMEPR bound. For further researches on Golay complementary sets and near-complementary sequences, see [5][15][20].

In this paper, we present a framework of near-complementary sequences of length $n2^{m-k}$ by the explicit Boolean expression and the equivalent array structure. While it transforms the seed pairs of length n to near-complementary sequences of length $n2^{m-k}$ by the aid of Golay complementary sequences of length 2^{m-k}, the framework preserves the PMEPR bound of the seed pairs. As the first example, we construct a new family of q-ary near-complementary sequences of length 2^m and PMEPR ≤ 4, where q is an even positive integer. In the sequence family, the framework employs a pair of Golay complementary sequences $-$ not necessarily a Golay complementary pair$-$ of length 2^k as the seed pair. We also obtain the explicit Boolean representation of the new sequence family, where it contains up to the third order coset of a q-ary generalization of the first order Reed-Muller code $\mathrm{RM}_q(1,m)$. In addition, we demonstrate how to generate the distinct sequences by developing a generation algorithm for the permutations in the Boolean expression. We show that the family contains more distinct sequences than any other known near complementary sequences of length 2^m and PMEPR ≤ 4.

As the next example, we present new families of q-ary near-complementary sequences of various lengths and PMEPR < 4. The framework constructs the sequence families of length $2^m \pm 2^{m-k}$, $2 \leq k \leq m-1$, by employing *shortened* and *extended* Golay complementary pairs as the seed pairs. In theoretical aspects, the sequence family of particular lengths can be an interesting example of sequences with low PMEPR for the lengths where no Golay complementary sequences exist [7]. Numerical results show that the new families produce a large number of distinct binary sequences for $5 \leq m \leq 7$, equivalent to a large number of distinct binary codewords with valid code rates. In practice, the sequence families can give the flexibility to coding solutions for peak power control in multicarrier communications by providing a variety of codewords with different lengths and low PMEPR.

The rest of this paper is organized as follows. Section II gives preliminaries for this work by describing definitions and notations throughout this paper. In Section III, we introduce a framework for near-complementary sequences by the explicit Boolean expression as well as the equivalent array structure. In Section IV, the framework constructs a new family of q-ary near-complementary sequences of length 2^m and PMEPR ≤ 4. We develop a generation algorithm for the permutations in the Boolean expression to obtain a large number of distinct sequences in the family. Section V presents new families of q-ary near-complementary sequences of various lengths and PMEPR < 4 by the framework employing shortened or extended Golay pairs. Numerical results show that the families provide a large number of distinct sequences for short lengths. Concluding remarks will be given in Section VI.

II. PRELIMINARIES

This section explains preliminaries for concepts and definitions in this work. The following notations will be used throughout this paper.

- q is an even positive integer.
- $\omega = e^{j\frac{2\pi}{q}}$ is a primitive q-th root of unity, where $j = \sqrt{-1}$.
- $\mathbb{Z}_q$ is a ring of integer modulo q.
- $\mathbb{Z}_q^m$ is an m-dimensional vector space where each component is an element in $\mathbb{Z}_q$.

A. Golay complementary sequences

Let $\mathbf{a} = (a_0, a_1, \cdots, a_{n-1})$ be a sequence over $\mathbb{Z}_q$ of length n. Then, the *aperiodic autocorrelation* of $\mathbf{a}$ is defined by

$$\rho_{\mathbf{a}}(\tau) = \sum_{i=0}^{n-1-\tau} \omega^{a_i - a_{i+\tau}}, \quad 0 \le \tau < n.$$

A q-ary sequence pair $(\mathbf{a}, \mathbf{b})$ of length n is called a *Golay complementary pair* if

$$\rho_{\mathbf{a}}(\tau) + \rho_{\mathbf{b}}(\tau) = 0 \text{ for all } \tau \neq 0 \tag{1}$$

where the sequence $\mathbf{a}$ (or $\mathbf{b}$) is called a *Golay complementary sequence*.

For a sequence $\mathbf{a} = (a_0, a_1, \cdots, a_{n-1})$, its *associated polynomial* $A(z)$ is defined by

$$A(z) = \sum_{i=0}^{n-1} \omega^{a_i} z^i. \tag{2}$$

In analysis of peak power control, it is convenient to assume that z in (2) lies on the unit circle in a complex plane, i.e., $z \in \{e^{j2\pi t} \mid 0 \le t < 1,\ j = \sqrt{-1}\}$. Then,

$$|A(z)|^2 = A(z)A^*(z) = n + \sum_{\tau=1}^{n-1} \rho_{\mathbf{a}}(\tau) z^{-\tau} + \sum_{\tau=1}^{n-1} \rho_{\mathbf{a}}^*(\tau) z^{\tau}, \quad |z| = 1 \tag{3}$$

where '*' denotes a complex conjugate. Assume $A(z)$ and $B(z)$ are the associated polynomials of a q-ary sequence pair $\mathbf{a}$ and $\mathbf{b}$ of length n, respectively. From (1), the sequence pair $(\mathbf{a}, \mathbf{b})$ forms a Golay complementary pair if

$$|A(z)|^2 + |B(z)|^2 = 2n. \tag{4}$$

For a brief overview to Golay complementary sequences, see [12].

B. *Peak power control in multicarrier communications*

In multicarrier communications with n subcarriers and q-PSK modulation, the transmitted signal for a q-ary sequence $\mathbf{a} = (a_0, \cdots, a_{n-1})$ can be modeled as the real part of

$$s_{\mathbf{a}}(t) = \sum_{i=0}^{n-1} \omega^{a_i} e^{j2\pi(f_0 + i\Delta f)t}, \quad t \in \left[0, \frac{1}{\Delta f}\right)$$

where Δf is the frequency separation between adjacent subcarriers and f_0 is the carrier frequency. Then, the *peak-to-mean envelope power ratio (PMEPR)* of $s_{\mathbf{a}}(t)$ is determined [10] by

$$\mathrm{PMEPR}(\mathbf{a}) = \frac{1}{n} \cdot \max_{t \in [0,1)} |A(e^{j2\pi t})|^2 \tag{5}$$

where $A(e^{j2\pi t})$ is the associated polynomial of $\mathbf{a}$ at $z = e^{j2\pi t}$. From (4) and (5), it is immediate that if $\mathbf{a}$ is a Golay complementary sequence, then $\mathrm{PMEPR}(\mathbf{a}) \leq 2$ since $|A(z)|^2 \leq 2n$ for any z with $|z| = 1$.

In general, if $\mathbf{a}$ is a q-ary sequence of length n and $|A(z)|^2 \leq vn$ for any z with $|z| = 1$, then $\mathrm{PMEPR}(\mathbf{a}) \leq v$. This implies that the study of associated polynomials and aperiodic autocorrelations plays a crucial role in PMEPR of sequences for multicarrier communications.

C. *Generalized Boolean functions*

Let $\mathbf{x} = (x_0, \cdots, x_{m-1})$ be a vector in $\mathbb{Z}_2^m$. A *generalized Boolean function* $f(\mathbf{x})$ is defined by a mapping $f : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_q$, which is represented by the sum of all possible products of x_l's with coefficients in $\mathbb{Z}_q$, i.e.,

$$f(\mathbf{x}) = f(x_0, \cdots, x_{m-1}) = \sum_{i=0}^{2^m-1} c_i \prod_{l=0}^{m-1} x_l^{i_l} \tag{6}$$

where $c_i \in \mathbb{Z}_q$ and $i_l \in \mathbb{Z}_2$ is obtained by a binary representation of $i = \sum_{l=0}^{m-1} i_l 2^l$. Note that the addition in the generalized Boolean function is computed modulo-q. (6) is also called the *algebraic normal form* of f. In (6), the order of the ith monomial with nonzero c_i is given by $\sum_{l=0}^{m-1} i_l$, and the highest order of monomials with nonzero c_i's is called the *(algebraic) degree* of the Boolean function f, denoted by $\deg(f)$.

Associated with a generalized Boolean function f, a sequence (or codeword) over $\mathbb{Z}_q$ of length 2^m can be given by $\mathbf{f} = (f_0, f_1, \cdots, f_{2^m-1})$, where $f_j = f(j_0, j_1, \cdots, j_{m-1}),\ j = \sum_{l=0}^{m-1} j_l 2^l \in \mathbb{Z}_{2^m}$ and $j_l \in \mathbb{Z}_2$. Hence, the *associated sequence* $\mathbf{f}$ of length 2^m is obtained by the generalized Boolean function f_j while j runs through 0 to $2^m - 1$ in the increasing order.

In this paper, we generalize the concept of the associated sequences so as to associate a sequence of arbitrary length with a generalized Boolean function.

Definition 1: Let m and n be positive integers, $2^{m-1} < n \le 2^m$. Let f be a generalized Boolean function of m variables. Let $\mathbb{X} = \{i_0, \cdots, i_{n-1}\} \subseteq \mathbb{Z}_{2^m}$ be a set of index $i_j,\ 0 \le j \le n-1$, where $i_0 < i_1 < \cdots < i_{n-1}$. Consider a sequence $\mathbf{f}$ over $\mathbb{Z}_q$ of length n, i.e.,

$$\mathbf{f} = (f_{i_0}, f_{i_1}, \cdots, f_{i_{n-1}}) \text{ where } f_{i_j} = f(i_{j_0}, i_{j_1}, \cdots, i_{j_{m-1}})$$

where $(i_{j_0}, i_{j_1}, \cdots, i_{j_{m-1}})$ is a binary representation of $i_j = \sum_{l=0}^{m-1} i_{j_l} 2^l,\ i_{j_l} \in \mathbb{Z}_2$. In other words, $\mathbf{f}$ is generated by the generalized Boolean function f_{i_j} while i_j runs through the elements in $\mathbb{X}$ as the increasing order. Then we say that the sequence $\mathbf{f}$ of length n is associated with the generalized Boolean function f.

From Definition 1, the conventional associated sequence of length 2^m is a special case of $n = 2^m$ and $\mathbb{X} = \mathbb{Z}_{2^m}$.

D. Reed-Muller codes

Let $\mathcal{C}$ be a linear code over $\mathbb{Z}_q$, where each codeword $\mathbf{c}$ of length n is a vector in $\mathbb{Z}_q^n$. For a vector $\mathbf{a}$ in $\mathbb{Z}_q^n$, the *coset* of $\mathcal{C}$ is defined by

$$\mathbf{a} + \mathcal{C} = \{\mathbf{a} + \mathbf{c} \mid \mathbf{c} \in \mathcal{C}\}$$

where $\mathbf{a}$ is called the *coset representative*.

For $q \ge 2$ and $0 \le r \le m$, the *generalized r-th order Reed-Muller code* $\mathrm{RM}_q(r, m)$ is defined by the set of sequences (or codewords) $\mathbf{f}$ of length 2^m associated with f, where f is a generalized Boolean function of degree at most r [6][16]. The classical binary Reed-Muller codes [11], denoted by $\mathrm{RM}(r, m)$, is thus a special case of $q = 2$. In [6], Davis and Jedwab showed that the

Golay complementary sequences of length 2^m can be obtained by the second order cosets of the generalized first order Reed-Muller codes $\mathrm{RM}_{2^h}(1,m)$ for $h \geq 1$. Paterson [16] then generalized the alphabet size to an even positive integer q.

Theorem 1: [16] Let q be even and π be a permutation in $\{0,1,\cdots,m-1\}$. Let $f:\mathbb{Z}_2^m \rightarrow \mathbb{Z}_q$ be a generalized Boolean function defined by

$$f(x_0,\cdots,x_{m-1}) = \frac{q}{2}\sum_{i=0}^{m-2} x_{\pi(i)}x_{\pi(i+1)} + \sum_{i=0}^{m-1} u_i x_i + e, \quad u_i,\ e \in \mathbb{Z}_q. \tag{7}$$

Then, the sequence $\mathbf{f}$ of length 2^m associated with f is a Golay complementary sequence.

Theorem 1 constructs a set of $(m!/2)\cdot q^{m+1}$ distinct Golay complementary sequences of length 2^m from the $(m!/2)$ second order cosets of $\mathrm{RM}_q(1,m)$ [16]. The Golay complementary sets and the near-complementary sequences of length 2^m in [5][15][16][19][20] are also represented as the second or higher order cosets of $\mathrm{RM}_q(1,m)$.

III. A FRAMEWORK FOR NEAR-COMPLEMENTARY SEQUENCES

This section introduces a framework that produces a large number of near-complementary sequences. The framework is given as the explicit Boolean expression and the equivalent array structure. The following notations will be used throughout this section.

- k and m are nonnegative integers with $0 \leq k \leq m-1$.
- $g_1, g_2 : \mathbb{Z}_2^k \rightarrow \mathbb{Z}_q$ are a pair of generalized Boolean functions of k variables.
- $\mathbf{g}_1$ and $\mathbf{g}_2$ are the sequences of length n, $2^{k-1} < n \leq 2^k$, associated with g_1 and g_2, respectively.
- $G_1(z)$ and $G_2(z)$ are the associated polynomials of sequences $\mathbf{g}_1$ and $\mathbf{g}_2$, respectively.

A. Basic framework

Generalizing the Rudin-Shapiro extension, Parker and Tellambura [13] presented a primitive framework for near-complementary sequences employing a seed pair, where the PMEPR bound of the seed sequences is preserved through the extension. Based on the work, Schmidt [19] gave a framework of the generalized Boolean function for near-complementary sequences of length 2^m, which is represented by a coset of $\mathrm{RM}_q(1,m)$. In [8], a framework of Golay complementary sequences of length $n2^{m-k}$ was presented by a $2^t \times n2^{m-k-t}$ matrix, where $0 \leq t \leq m-k$, employing the seed pairs of length n and the controlling pairs of length 2^{m-k}. If $n = 2^k$, the framework is then given by the algebraic normal form in Lemma 9 of [8].

In the following, we present a generalized framework for near-complementary sequences of length $n2^{m-k}$ by integrating the previous works.

Theorem 2: For $0 \leq t \leq m-k$, let π be a permutation in a nonnegative integer set $\Omega = \{0, 1, \cdots, t-1, t+k, \cdots, m-1\}$. Let $f^{(t)}$ be a generalized Boolean function defined by

$$\begin{aligned} & f^{(t)}(x_0, \cdots, x_{m-1}) \\ & = \frac{q}{2}\sum_{r=0}^{t-2} x_{\pi(r)}x_{\pi(r+1)} + \frac{q}{2}x_{\pi(t-1)}x_{\pi(t+k)} + \frac{q}{2}\sum_{r=t+k}^{m-2} x_{\pi(r)}x_{\pi(r+1)} + \sum_{r\in\Omega} u_r x_r + e \\ & + x_{\pi(\phi)} \cdot (-g_1(x_t, \cdots, x_{t+k-1}) + g_2(x_t, \cdots, x_{t+k-1})) + g_1(x_t, \cdots, x_{t+k-1}). \end{aligned} \tag{8}$$

where $u_r,\ e \in \mathbb{Z}_q$. In (8), $\phi = k$ if $t = 0$, and $\phi = 0$ otherwise. Note each term in (8) exists only with the valid subscripts and arguments. As $(x_t, \cdots, x_{t+k-1})$ runs through $(0, \cdots, 0)$ to $(n_0, \cdots, n_{k-1})$, the generalized Boolean functions g_1 and g_2 generate the seed pair $\mathbf{g}_1$ and $\mathbf{g}_2$ of length n, $2^{k-1} < n \leq 2^k$, respectively, where $n-1 = \sum_{l=0}^{k-1} n_l 2^l,\ n_l \in \mathbb{Z}_2$. Let a q-ary sequence $\mathbf{f}^{(t)} = (f_0, f_1, \cdots, f_{n2^{m-k}-1})$ of length $n2^{m-k}$ be generated by $f^{(t)}$ while $x = \sum_{l=0}^{m-1} x_l 2^l$ runs through the elements in $\mathbb{X}_t = \{x \mid x \in \mathbb{Z}_{2^m} \text{ where } 0 \leq \sum_{l=0}^{k-1} x_{t+l}2^l \leq n-1\}$ in the increasing order. Then, the PMEPR of $\mathbf{f}^{(t)}$ is bounded by

$$\mathrm{PMEPR}(\mathbf{f}^{(t)}) \leq \frac{|G_1(z)|^2 + |G_2(z)|^2}{n}. \tag{9}$$

We have to mention that the framework for $t = 0$ is equivalent to the one in [13][14]. for near-complementary sequences of length $n2^{m-k}$. In fact, the framework is a generalization of the results in [13] and [19], motivated by [8]. Before proving Theorem 2, we describe the array structure equivalent to the framework.

B. Equivalent array structure

Given the notations in Theorem 2, Table I displays the arrangement of $\mathbf{f}^{(t)}$ in a $2^{m-k} \times n$ array, where each subscript of f is $x = \sum_{l=0}^{m-1} x_l 2^l \in \mathbb{X}_t$. Apparently, reading the array as the increasing order of x is equivalent to choosing each $2^t \times n$ sub-matrix from top to bottom and reading it column-wise.

TABLE I

THE ARRAY STRUCTURE OF $\mathbf{f}^{(t)}$

						x_{t+k-1}	0	0	$\cdots$	n_{k-1}	
						$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	
						x_t	0	1	$\cdots$	n_0	
x_{m-1}	$\cdots$	x_{t+k}	x_{t-1}	$\cdots$	x_0						
0	$\cdots$	0	0	$\cdots$	0		f_0	f_{2^t}	$\cdots$	$f_{2^t n-2^t}$	$\mathbf{s}_0$
0	$\cdots$	0	0	$\cdots$	1		f_1	f_{2^t+1}	$\cdots$	$f_{2^t n-2^t+1}$	$\mathbf{s}_1$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$		$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
0	$\cdots$	0	1	$\cdots$	1		f_{2^t-1}	$f_{2^{t+1}-1}$	$\cdots$	$f_{2^t n-1}$	$\mathbf{s}_{2^t-1}$
0	$\cdots$	1	0	$\cdots$	0		$f_{2^{t+k}}$	$f_{2^{t+k}+2^t}$	$\cdots$	$f_{2^{t+k}+2^t n-2^t}$	$\mathbf{s}_{2^t}$
0	$\cdots$	1	0	$\cdots$	1		$f_{2^{t+k}+1}$	$f_{2^{t+k}+2^t+1}$	$\cdots$	$f_{2^{t+k}+2^t n-2^t+1}$	$\mathbf{s}_{2^t+1}$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$		$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
0	$\cdots$	1	1	$\cdots$	1		$f_{2^{t+k}+2^t-1}$	$f_{2^{t+k}+2^{t+1}-1}$	$\cdots$	$f_{2^{t+k}+2^t n-1}$	$\mathbf{s}_{2^{t+1}-1}$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$		$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
1	$\cdots$	1	0	$\cdots$	0		$f_{2^m-2^{t+k}}$	$f_{2^m-2^{t+k}+2^t}$	$\cdots$	$f_{2^m-(2^k-n+1)2^t}$	$\mathbf{s}_{2^{m-k}-2^t}$
1	$\cdots$	1	0	$\cdots$	1		$f_{2^m-2^{t+k}+1}$	$f_{2^m-2^{t+k}+2^t+1}$	$\cdots$	$f_{2^m-(2^k-n+1)2^t+1}$	$\mathbf{s}_{2^{m-k}-2^t+1}$
$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$		$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
1	$\cdots$	1	1	$\cdots$	1		$f_{2^m-2^{t+k}+2^t-1}$	$f_{2^m-2^{t+k}+2^{t+1}-1}$	$\cdots$	$f_{2^m-(2^k-n)2^t-1}$	$\mathbf{s}_{2^{m-k}-1}$

In what follows, we investigate the array by slightly modifying its row index. The modification helps us to prove Theorem 2 and to use the array in a simpler manner for near-complementary sequences. In the array of Table I, each row vector $\mathbf{s}_i,\ 0 \le i \le 2^{m-k}-1$, is obtained when the row index $(x_0, \cdots, x_{t-1}, x_{t+k}, \cdots, x_{m-1})$ is given as a constant vector. From (8), we see

$$\mathbf{s}_i = \begin{cases} \mathbf{g}_1 + c_i \cdot \mathbf{1}, & \text{if } x_{\pi(\phi)} = 0, \\ \mathbf{g}_2 + c_i \cdot \mathbf{1}, & \text{if } x_{\pi(\phi)} = 1 \end{cases} \tag{10}$$

where $\mathbf{1} = (1, 1, \cdots, 1)$ of length n and $c_i \in \mathbb{Z}_q$. In (10), note that the addition is computed modulo-q. The successive increase of the row index allows its conversion $(x_0, \cdots, x_{t-1}, x_{t+k}, \cdots, x_{m-1}) \to (x_0, \cdots, x_{m-k-1})$ by changing x_r to x_{r-k} for $r \in \Omega \ge t+k$. Then, the permutation π in Ω induces its equivalence π' in $\{0, \cdots, m-k-1\}$, where

$$\pi(r) = \begin{cases} \pi'(j), & \text{if } \pi'(j) \le t-1 \\ \pi'(j) + k, & \text{if } \pi'(j) \ge t \end{cases}$$

where $j = r$ if $r \leq t-1$ and $j = r-k$ if $r \geq t+k$. From the equivalence and the index conversion, it is easy to show that $\pi'(0)$ can replace $\pi(\phi)$ in (10). With the new row index $(x_0, \cdots, x_{m-k-1})$, it is apparent that c_i is the ith element of a Golay complementary sequence $\mathbf{c}$ of length 2^{m-k} associated with a generalized Boolean function of $(m-k)$ variables with the permutation π', i.e.,

$$c(x_0, \cdots, x_{m-k-1}) = \frac{q}{2} \sum_{r=0}^{m-k-2} x_{\pi'(r)} x_{\pi'(r+1)} + \sum_{r=0}^{m-k-1} u_r x_r + e, \quad u_r, e \in \mathbb{Z}_q.$$

Ultimately, the array structure is equivalent to the arrangement of the seed pair $\mathbf{g}_1$ and $\mathbf{g}_2$ in each row, controlled by $x_{\pi'(0)}$ and $\mathbf{c}$. Finally, choosing each $2^t \times n$ sub-matrix from top to bottom and reading it column-wise then produce $\mathbf{f}^{(t)}$.

We are now ready to prove Theorem 2.

Proof of Theorem 2: Applying the coordinate variation of $(x_0, \cdots, x_{k-1}) \rightarrow (x_t, \cdots, x_{t+k-1})$ to the framework in Corollary 15 of [19], we obtain $f^{(t)}$ in (8). Controlled by $x_{\pi'(0)}$ and $\mathbf{c}$ in the array, $\mathbf{f}^{(t)}$ is ultimately equivalent to the sequence generated by the matrix M of Lemma 5 in [8], where the seeds are replaced by $\mathbf{g}_1$ and $\mathbf{g}_2$. Similar to the proof of the lemma, it is immediate that the PMEPR of $\mathbf{f}^{(t)}$ is given by

$$\mathrm{PMEPR}(\mathbf{f}^{(t)}) \leq \frac{2^{m-k} \cdot (|G_1(z^{2^t})|^2 + |G_2(z^{2^t})|^2)}{n 2^{m-k}} = \frac{|G_1(z)|^2 + |G_2(z)|^2}{n}.$$

□

Corollary 1: In Theorem 2, let g_1 and g_2 generate Golay complementary sequences $\mathbf{g}_1$ and $\mathbf{g}_2$, respectively, where $\mathbf{g}_1$ and $\mathbf{g}_2$ are not necessarily a Golay complementary pair. Then, the PMEPR of $\mathbf{f}^{(t)}$ is at most 4.

Proof: If $(\mathbf{g}_1, \mathbf{g}_2)$ is a Golay complementary pair, then

$$|G_1(z)|^2 + |G_2(z)|^2 = 2n. \tag{11}$$

On the other hand, if $(\mathbf{g}_1, \mathbf{g}_2)$ is not a Golay complementary pair, then

$$|G_1(z)|^2 \leq 2n \text{ and } |G_2(z)|^2 \leq 2n \tag{12}$$

as they are still Golay complementary sequences. From (11), (12) and (9), $\mathrm{PMEPR}(\mathbf{f}) \leq 4$. □

If we consider $m-k$ as a single variable v (≥ 1), then the $2^v \times n$ array with the row index $(x_0, \cdots, x_{v-1})$ can be used for constructing near-complementary sequences. In the following, we describe a procedure to transform the seed pair $\mathbf{g}_1$ and $\mathbf{g}_2$ of length n to near-complementary sequences $\mathbf{f}^{(t)}$ of length $n2^v$, preserving the PMEPR bound of the seed pair.

Procedure:

Input: a seed pair $\mathbf{g}_1$ and $\mathbf{g}_2$ of length n.

Output: a near-complementary sequence $\mathbf{f}^{(t)}$ of length $n2^v$, where $0 \leq t \leq v$.

1. Arrange the row index $(x_0, \cdots, x_{v-1})$ as the increasing order of $i = \sum_{l=0}^{v-1} x_l 2^l$ in a $2^v \times n$ array, and generate a permutation π' in $\{0, \cdots, v-1\}$.
2. Transform the seeds by (10) with a Golay complementary sequence $\mathbf{c} = (c_0, \cdots, c_{2^v-1})$ with the permutation π', where $\pi'(0)$ replaces $\pi(\phi)$ in (10). Place the transformed seed $\mathbf{s}_i$ in the ith row of the array for $0 \leq i \leq 2^v - 1$.
3. Choose a $2^t \times n$ sub-matrix corresponding to constant $(x_t, \cdots, x_{v-1})$, and read it column-wise from left to right. Repeat for all the sub-matrices from top to bottom.

Example 1: In the above procedure, let $n = v = 3$. Consider the permutation $\pi' : (0,1,2) \rightarrow (0,2,1)$, where $\pi'(0) = 0$. Let the Golay complementary sequence $\mathbf{c}$ be generated by $c(x_0, x_1, x_2) = x_0x_2 + x_1x_2 + x_0$. Consider a seed pair given by $\mathbf{g}_1 = (0,1,1)$ and $\mathbf{g}_2 = (0,0,1)$, where $\text{PMEPR}(\mathbf{g}_1) = \text{PMEPR}(\mathbf{g}_2) = 1.667$. Then, we can establish the 8×3 array as in Table II, where $\overline{\mathbf{g}}_j = \mathbf{g}_j + (1,1,1),\ j = 1,2$. Note that the selection of $\mathbf{g}_1$ or $\mathbf{g}_2$ in each row is controlled by $x_{\pi'(0)} = x_0$. On the other hand, c_i selects $\mathbf{g}_j$ (when $c_i = 0$) or $\overline{\mathbf{g}}_j$ (when $c_i = 1$). If $t = 2$, we first choose the upper 4×3 sub-matrix and read it column-wise. Then, the lower 4×3 sub-matrix is chosen and read column-wise. Finally, the near-complementary sequences of length $2^v \times n = 24$ is given by $\mathbf{f}^{(2)} = (0,1,0,1,1,1,1,1,1,0,1,0,0,0,1,1,1,0,0,1,1,1,0,0)$ where $\text{PMEPR}(\mathbf{f}^{(2)}) = 2.820$. From the similar approach, $\mathbf{f}^{(1)} = (0,1,1,1,1,0,0,1,1,1,1,0,0,0,1,0,1,1,1,1,0,1,0,0)$ where $\text{PMEPR}(\mathbf{f}^{(1)}) = 3.317$, and $\mathbf{f}^{(0)} = (0,1,1,1,1,0,0,1,1,1,1,0,0,1,1,0,0,1,1,0,0,1,1,0)$ where $\text{PMEPR}(\mathbf{f}^{(0)}) = 3.255$. □

Note the procedure generates a variety of near-complementary sequences $\mathbf{f}^{(t)}$ of length $n2^v$ by varying t for different seed pairs $(\mathbf{g}_1, \mathbf{g}_2)$ and different Golay sequences $\mathbf{c}$.

In summary, the near-complementary sequences of length $n2^v$ can be obtained by the *transformation* of the seed pairs of length n by Golay complementary sequences of length 2^v. From the equivalence to Theorem 2, the sequences has the same PMEPR bound as the seeds in (9).

TABLE II

THE ARRAY STRUCTURE OF EXAMPLE 1

$x_2x_1x_0$	$\mathbf{c}$	$\mathbf{s}_i$	$\mathbf{f}^{(t)}$
000	0	$\mathbf{g}_1$	011
001	1	$\overline{\mathbf{g}}_2$	110
010	0	$\mathbf{g}_1$	011
011	1	$\overline{\mathbf{g}}_2$	110
100	0	$\mathbf{g}_1$	011
101	0	$\mathbf{g}_2$	001
110	1	$\overline{\mathbf{g}}_1$	100
111	1	$\overline{\mathbf{g}}_2$	110

IV. DISTINCT NEAR COMPLEMENTARY SEQUENCES OF LENGTH 2^m AND PMEPR ≤ 4

A. *Construction*

In Theorem 2, we employ a pair of Golay complementary sequences of length 2^k − not necessarily a Golay complementary pair − as the seeds in the framework for constructing a new family of near-complementary sequences of length 2^m and PMEPR ≤ 4.

Construction 1: For a positive integer $m \geq 3$, let $0 \leq k \leq m-1$ and $0 \leq t \leq m-k$. Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be Golay complementary sequences of length 2^k associated with the generalized Boolean functions of k variables as in (7). Employing $\mathbf{g}_1$ and $\mathbf{g}_2$ as the seeds, the framework of Theorem 2 presents the Boolean representation of

$$\begin{aligned} f^{(t)}(x_0,\cdots,x_{m-1}) =& \frac{q}{2}\sum_{r=0}^{t-2} x_{\pi(r)}x_{\pi(r+1)} + \frac{q}{2}x_{\pi(t-1)}x_{\pi(t+k)} + \frac{q}{2}\sum_{r=t+k}^{m-2} x_{\pi(r)}x_{\pi(r+1)} \\ &+ \frac{q}{2}x_{\pi(\phi)} \cdot \sum_{r=t}^{t+k-2} \left(-x_{\pi_1(r)}x_{\pi_1(r+1)} + x_{\pi_2(r)}x_{\pi_2(r+1)}\right) \\ &+ x_{\pi(\phi)} \cdot \sum_{r=t}^{t+k-1} v_r x_r + \frac{q}{2}\sum_{r=t}^{t+k-2} x_{\pi_1(r)}x_{\pi_1(r+1)} + \sum_{r=0}^{m-1} u_r x_r + e \end{aligned} \tag{13}$$

where $u_r, v_r, e \in \mathbb{Z}_q$, and $\phi = k$ if $t = 0$, and $\phi = 0$ otherwise. In (13), π_1 and π_2 are permutations in $\Omega_0 = \{t, t+1, \cdots, t+k-1\}$, while π is a permutation in $\Omega = \{0, \cdots, t-1, t+k, \cdots, m-1\}$. Then, $f^{(t)}$ generates the associated sequences $\mathbf{f}^{(t)}$ of length 2^m and PMEPR ≤ 4 while x runs through all elements in $\mathbb{Z}_{2^m}$ as the increasing order.

In Construction 1, we can generate a variety of the near-complementary sequences while k runs through 0 to $m-1$ and t through 0 to $m-k$ for a given k. Lemma 1 discusses the maximum number of distinct sequences in the family.

Lemma 1: For a given k, the number of the distinct near-complementary sequences in Construction 1 is at most N_k, where

$$N_k = \begin{cases} \frac{m!}{2} \cdot q^{m+1}, & k = 0, \\ m! \cdot q^{m+2}, & k = 1, \\ \frac{(k!)^2 (m-k+1)!}{4} \cdot q^{m+k+1}, & 2 \le k \le m-2, \\ \frac{((m-1)!)^2}{2} \cdot q^{2m}, & k = m-1. \end{cases}$$

Thus, the number of the distinct near-complementary sequences for all k's is at most $N_{\max} = \sum_{k=0}^{m-1} N_k$.

Proof: In (13), it is obvious that the reverse ordering of permutations π_1 and π_2 must be excluded to avoid multiple sequences. On the other hand, the reverse ordering of π is allowed since $x_{\pi(\phi)}$ is connected to x_r's, $r \in \Omega_0$. Including t running through 0 to $m-k$ for a given k,

$$\begin{aligned} N_k &= \frac{k!}{2} \cdot \frac{k!}{2} \cdot (m-k)! \cdot (m-k+1) \cdot q^{m+k+1} \\ &= \frac{(k!)^2 (m-k+1)!}{4} \cdot q^{m+k+1}, \quad 2 \le k \le m-2. \end{aligned}$$

If $k = 1$, the number of the valid permutations of π_1 and π_2 is simply one, thus $N_1 = 1 \cdot 1 \cdot (m-1)! \cdot m \cdot q^{m+2} = m! \cdot q^{m+2}$. Similarly, if $k = m-1$, then $N_{m-1} = \frac{(m-1)!}{2} \cdot \frac{(m-1)!}{2} \cdot 1! \cdot 2 \cdot q^{2m} = \frac{((m-1)!)^2}{2} \cdot q^{2m}$. Finally, if $k = 0$, then $f^{(t)}$ in (13) becomes the Boolean expression of Golay complementary sequences regardless of t. Therefore, $N_0 = \frac{m!}{2} \cdot q^{m+1}$. $\square$

In counting the maximum number of the distinct near-complementary sequences, we still have the possibility that some sequences are generated more than once. In fact, we observed the multiple generation from our experiments. Therefore, we need to consider how to generate distinct sequences in the family. Lemma 1 convinces us that most of the near-complementary sequences are produced at $k = m-1$. In the following subsections, we thus restrict our attention to the case of $k = m-1$ instead of all k's, and describe an approach for generating as many distinct sequences as possible in a simple and constructive way.

B. Sufficient condition for distinct sequences

For $k = m-1$, $t = 0$ or 1, and

$$\begin{aligned} f^{(0)}(x_0, \cdots, x_{m-1}) &= \frac{q}{2} x_{m-1} \cdot \sum_{r=0}^{m-3} \left(-x_{\pi'_1(r)} x_{\pi'_1(r+1)} + x_{\pi'_2(r)} x_{\pi'_2(r+1)} \right) \\ &\qquad + \frac{q}{2} \sum_{r=0}^{m-3} x_{\pi'_1(r)} x_{\pi'_1(r+1)} + x_{m-1} \cdot \sum_{r=0}^{m-2} v'_r x_r + \sum_{r=0}^{m-1} u'_r x_r + e', \\ f^{(1)}(x_0, \cdots, x_{m-1}) &= \frac{q}{2} x_0 \cdot \sum_{r=1}^{m-2} \left(-x_{\pi_1(r)} x_{\pi_1(r+1)} + x_{\pi_2(r)} x_{\pi_2(r+1)} \right) \\ &\qquad + \frac{q}{2} \sum_{r=1}^{m-2} x_{\pi_1(r)} x_{\pi_1(r+1)} + x_0 \cdot \sum_{r=1}^{m-1} v_r x_r + \sum_{r=0}^{m-1} u_r x_r + e \end{aligned} \tag{14}$$

where '$'$' is to distinguish $f^{(0)}$ and $f^{(1)}$. If $m \geq 3$, all possible sequences associated with $f^{(t)}$ are distinct for a given t due to the existence of the second order coset representatives. However, we have the potential that a sequence in $f^{(0)}$ is identical to the one in $f^{(1)}$. Now, we consider the necessary condition for the case.

Lemma 2: Let s_0 and s_1 be the third order coset representatives in $f^{(0)}$ and $f^{(1)}$, i.e.,

$$\begin{aligned} s_0(x_0, \cdots, x_{m-1}) &= \frac{q}{2} x_{m-1} \cdot \sum_{r=0}^{m-3} \left(-x_{\pi'_1(r)} x_{\pi'_1(r+1)} + x_{\pi'_2(r)} x_{\pi'_2(r+1)} \right) \\ s_1(x_0, \cdots, x_{m-1}) &= \frac{q}{2} x_0 \cdot \sum_{r=1}^{m-2} \left(-x_{\pi_1(r)} x_{\pi_1(r+1)} + x_{\pi_2(r)} x_{\pi_2(r+1)} \right). \end{aligned} \tag{15}$$

If $f^{(1)}$ generates the same sequence as $f^{(0)}$, then $s_0 = s_1$ and

$$s_1(x_0, \cdots, x_{m-1}) = \frac{q}{2} x_0 x_{m-1} \cdot \hat{s}(x_1, \cdots, x_{m-2}) \tag{16}$$

where $\hat{s}(x_1, \cdots, x_{m-2}) = \sum_{r=1}^{m-2} w_r x_r,\ w_r \in \mathbb{Z}_2$.

Proof: As s_0 and s_1 are the unique third order coset representatives in $f^{(0)}$ and $f^{(1)}$, respectively, it is obvious that if $s_0 \neq s_1$, then $f^{(0)}$ and $f^{(1)}$ always generate distinct sequences. The inverse induces the necessary condition. For $s_0 = s_1$ in (15), only the monomials with x_0 must remain in the quadratic sum of s_0, and simultaneously the surviving monomials in the quadratic sum of s_1 must have x_{m-1}. Therefore, $s_0 = s_1$ results in (16). □

From Lemma 2, the sufficient condition for distinct sequences from $f^{(0)}$ and $f^{(1)}$ is obvious; If $s_0 \neq \frac{q}{2} x_0 x_{m-1} \cdot \hat{s}$ or $s_1 \neq \frac{q}{2} x_0 x_{m-1} \cdot \hat{s}$, then $f^{(0)} \neq f^{(1)}$, i.e., the associated sequences are

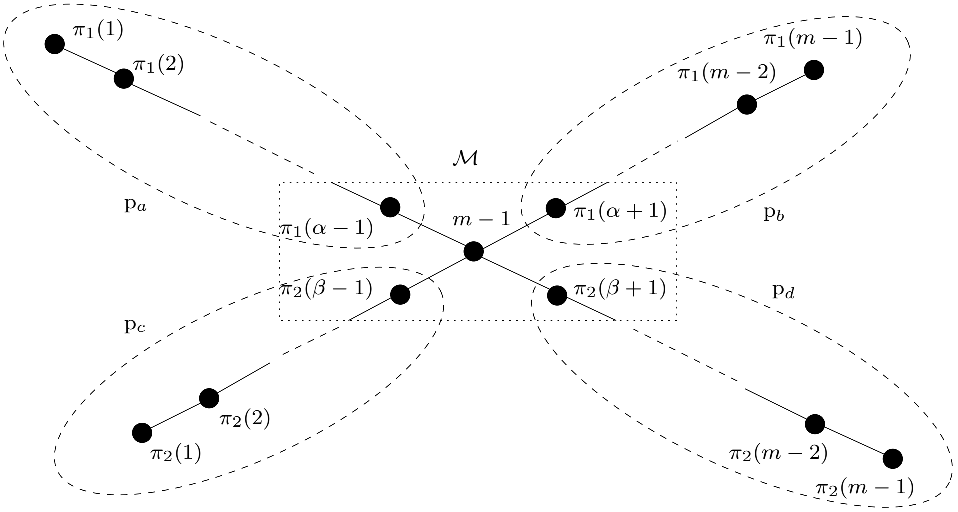

Fig. 1. The graph of the quadratic form Q_1

distinct. In our approach, we make no restrictions on π_1' and π_2', so we have the potential of $s_0 = \frac{q}{2} x_0 x_{m-1} \cdot \hat{s}$ at $t = 0$. Instead, we restrict a permutation pattern π_2 at $t = 1$ such that it does not cause (16).

Lemma 3: At $t = 1$, let π_1 be a permutation in $\Omega_0 = \{1, 2, \cdots, m-1\}$, denoted by $\pi_1 = (\mathbf{p}_a, m-1, \mathbf{p}_b)$, where $\mathbf{p}_a$ and $\mathbf{p}_b$ are disjoint sets such that $\mathbf{p}_a \cup \mathbf{p}_b = \Omega_0 \setminus \{m-1\}$. Let $\pi_1(\alpha) = m-1$ (or $\alpha = \pi_1^{-1}(m-1)$) for $1 \le \alpha \le m-1$. Then, π_1 and π_2 cause (16) for s_1 if and only if π_2 belongs to the following *equivalent* patterns,

$$\begin{aligned}
&(\mathbf{p}_a, m-1, \mathbf{p}_b),\ (\overleftarrow{\mathbf{p}_a}, m-1, \mathbf{p}_b), (\mathbf{p}_a, m-1, \overleftarrow{\mathbf{p}_b}),\ (\overleftarrow{\mathbf{p}_a}, m-1, \overleftarrow{\mathbf{p}_b}), \\
&(\mathbf{p}_b, m-1, \mathbf{p}_a),\ (\mathbf{p}_b, m-1, \overleftarrow{\mathbf{p}_a}), (\overleftarrow{\mathbf{p}_b}, m-1, \mathbf{p}_a),\ (\overleftarrow{\mathbf{p}_b}, m-1, \overleftarrow{\mathbf{p}_a})
\end{aligned} \tag{17}$$

where $\overleftarrow{\mathbf{p}_*}$ is the reverse ordering of $\mathbf{p}_*$. In the equivalent patterns, $\pi_2^{-1}(m-1) = \alpha$ in the first half, and $\pi_2^{-1}(m-1) = m - \alpha$ in the last half.

Proof: In (15), denote the quadratic sum of s_1 by

$$Q_1 = \frac{q}{2} \sum_{r=1}^{m-2} \left(-x_{\pi_1(r)} x_{\pi_1(r+1)} + x_{\pi_2(r)} x_{\pi_2(r+1)} \right).$$

Figure 1 illustrates the graph of the quadratic form Q_1. In the figure, $\pi_2 = (\mathbf{p}_c, m-1, \mathbf{p}_d)$ and $\pi_2(\beta) = m-1$, where $\mathbf{p}_c$ and $\mathbf{p}_d$ are disjoint sets such that $\mathbf{p}_c \cup \mathbf{p}_d = \Omega_0 \setminus \{m-1\}$. Here, we equivalently consider each element in π_1 and π_2 as the corresponding variable in Q_1. If (16) is achieved, then the variables not included in the set $\mathcal{M}$ must be canceled out in Q_1 so that the quadratic sum finally has a common variable x_{m-1}. It implies that the variables of $\mathbf{p}_c$ other than $\pi_2(\beta-1)$ should be canceled out by the variables from either $\mathbf{p}_a$ or $\mathbf{p}_b$, which results in $\mathbf{p}_c \in \{\mathbf{p}_a, \overleftarrow{\mathbf{p}_a}, \mathbf{p}_b, \overleftarrow{\mathbf{p}_b}\}$. Similarly, the variables of $\mathbf{p}_d$ other than $\pi_2(\beta+1)$ should be canceled out by the variables from either $\mathbf{p}_b$ or $\mathbf{p}_a$, which results in $\mathbf{p}_d \in \{\mathbf{p}_a, \overleftarrow{\mathbf{p}_a}, \mathbf{p}_b, \overleftarrow{\mathbf{p}_b}\}$. Note that if $\mathbf{p}_c = \mathbf{p}_a$ or $\overleftarrow{\mathbf{p}_a}$, then $\mathbf{p}_d = \mathbf{p}_b$ or $\overleftarrow{\mathbf{p}_b}$, and vice versa. Consequently, if (16) is achieved, then $\pi_2 = (\mathbf{p}_c, m-1, \mathbf{p}_d)$ is obviously equivalent to one of the above eight patterns in (17).

Conversely, if π_2 belongs to the above eight patterns, it is straightforward that s_1 achieves (16). In Figure 1, if we compare the number of variables, then $\beta = \pi_2^{-1}(m-1) = \alpha$ for $\mathbf{p}_c = \mathbf{p}_a$ or $\overleftarrow{\mathbf{p}_a}$. Similarly, $\beta = m - \alpha$ if $\mathbf{p}_c = \mathbf{p}_b$ or $\overleftarrow{\mathbf{p}_b}$. □

We clarify the sufficient condition in the following lemma.

Lemma 4: At $t=1$, let a permutation π_1 be given for $f^{(1)}$ in (14). If we prevent any equivalent pattern in (17) from arising in π_2, then the corresponding $f^{(1)}$ produces the sequences distinct from the ones associated with $f^{(0)}$ regardless of π_1' and π_2' at $t=0$.

Proof: From Lemmas 2 and 3, it is immediate. □

C. Permutation generation for distinct sequences

The sufficient condition for distinct sequences requires us to first generate all possible permutations π_1 and π_2, then compare them, and finally select the appropriate π_2 for preventing (16). However, this non-constructive method seems not efficient in time and complexity. Instead, we describe an efficient and constructive approach for the permutation π_2 to prevent (16), and present the algorithm for generating the pair π_1 and π_2.

At $t=1$, let π_1 be given as a permutation in Lemma 3. For π_2, we first develop a permutation $\widetilde{\pi}_2 = (\widetilde{\pi}_2(1), \cdots, \widetilde{\pi}_2(m-2))$ in $\widetilde{\Omega}_0 = \Omega_0 \setminus \{m-1\}$. Since $\widetilde{\pi}_2$ has $m-2$ elements in it, there are $m-1$ candidate positions to insert the index $m-1$ for completing π_2, where the candidate position index is denoted by $l,\ 1 \le l \le m-1$. If π_2 is one of the equivalent patterns in Lemma 3, $\pi_2^{-1}(m-1) = \alpha$ or $m-\alpha$. Therefore, if we exclude the indices of $l = \alpha$ and $l = m - \alpha$ from the candidate positions for inserting $m-1$, we prevent all the equivalent patterns in (17) for π_2. By inserting the index $m-1$ to the other possible positions in $\widetilde{\pi}_2$, we obtain a permutation π_2 that never produces the equivalent patterns and eventually prevents (16).

In what follows, we present a generation algorithm for π_1 and π_2 at $t = 1$ for $f^{(0)} \neq f^{(1)}$.

Permutation Generation at $t = 1$:

```
Initialization: Set $\Omega_0 = \{1, 2, \cdots, m-1\}$ and $\widetilde{\Omega}_0 = \{1, 2, \cdots, m-2\}$. Set $M_1 = \frac{(m-1)!}{2}$ and $M_2 = \frac{(m-2)!}{2}$.
Algorithm 1:
for $i$ from 1 to $M_1$
        Generate a permutation $\pi_{0,i}$ in $\Omega_0$ and set it to $\pi_1$.
        for $j$ from 1 to $M_2$
                Generate a permutation $\widetilde{\pi}_{0,j}$ in $\widetilde{\Omega}_0$ and set it to $\widetilde{\pi}_2$.
                for $l$ from 1 to $m-1$
                        if $l \neq \alpha$ and $l \neq m-\alpha$ then
                                Insert the index $m-1$ into a position $l$ in $\widetilde{\pi}_2$ and set it to $\pi_2$.
                                Form a permutation set $(\pi_1, \pi_2)$.
                                Return $(\pi_1, \pi_2)$.
```

Note that the permutations generated by the algorithm form a subset of all the permutations that can prevent (16). We now count the number of the possible permutations (π_1, π_2) generated by the algorithm.

Lemma 5: The number of the permutation pairs (π_1, π_2) generated by Algorithm 1 is

$$P_1 = \begin{cases} \frac{(m-1)!(m-2)!}{4} \cdot (m-3), & \text{for odd } m, \\ \left(\frac{(m-2)!}{2}\right)^2 \cdot (m-2)^2, & \text{for even } m. \end{cases} \tag{18}$$

Proof: If m is odd, then $\alpha \neq m - \alpha$, so there are always two different positions where the index $m-1$ must not be inserted. While i, j and l run through their entire values, P_1 is given by

$$P_1 = M_1 \cdot M_2 \cdot (m-3) = \frac{(m-1)!(m-2)!}{4} \cdot (m-3). \tag{19}$$

If m is even and $\alpha = m/2$, on the other hand, only one position must be excluded, and it is easy to see that the number of the case is $\frac{(m-2)!}{2}$. When $\alpha \neq m/2$ for even m, there are two different positions of α and $m-\alpha$ to be excluded. Thus, for even m, while i, j and l run through their entire values in the algorithm,

$$\begin{aligned} P_1 &= \left(M_1 - \frac{(m-2)!}{2}\right) \cdot M_2 \cdot (m-3) + \frac{(m-2)!}{2} \cdot M_2 \cdot (m-2) \\ &= \left(\frac{(m-2)!}{2}\right)^2 \cdot (m-2)^2. \end{aligned} \tag{20}$$

From (19) and (20), the proof is completed. □

D. Construction for distinct sequences

We summarize our construction for *distinct* near-complementary sequences.

Construction 2: For the construction of distinct near-complementary sequences of length 2^m and PMEPR ≤ 4, we consider the special case of Construction 1.

- In Construction 1, set $k = m - 1$. Accordingly, $t = 0$ and 1 in the construction.
- Use the Boolean functions $f^{(0)}$ and $f^{(1)}$ in (14), where u'_r, u_r, v'_r, v_r, e', and e can be any element in $\mathbb{Z}_q$.
- In $f^{(0)}$, all possible permutation pairs (π'_1, π'_2) are allowed. (including $\pi'_1 = \pi'_2$.)
- In $f^{(1)}$, the permutation pair (π_1, π_2) is generated by Algorithm 1.

We are now ready to enumerate the distinct sequences in our near-complementary sequence family.

Lemma 6: The total number of distinct near-complementary sequences in Construction 2 is

$$N_{\text{dist}} = \begin{cases} \frac{(m-1)!}{2} \cdot (m-2)! \cdot (m-2) \cdot q^{2m}, & \text{for odd } m, \\ \left(\frac{(m-2)!}{2}\right)^2 \cdot (2m^2 - 6m + 5) \cdot q^{2m}, & \text{for even } m. \end{cases} \tag{21}$$

Proof: If $t = 0$, the number of permutations is $P_0 = \left(\frac{(m-1)!}{2}\right)^2$. If $t = 1$, on the other hand, the number of permutations P_1 is given by (18). Thus, the total number of distinct sequences is $N_{\text{dist}} = (P_0 + P_1) \cdot q^{2m}$. By applying P_0 and P_1, we obtain (21). □

Table III shows the code rates of the binary $(q = 2)$ and the quaternary $(q = 4)$ near-complementary sequences of length 2^m and PMEPR ≤ 4, where we consider each sequence as a codeword in a coset of $\text{RM}_q(1, m)$. In the table, $R_{\max} = (\log_q N_{\max})/2^m$ is the code rate of the maximum possible distinct sequences in Construction 1, where $N_{\max}$ is given in Lemma 1.

TABLE III

THE CODE RATES FOR NEAR-COMPLEMENTARY SEQUENCES OF LENGTH 2^m AND PMEPR ≤ 4

m	Binary		Quaternary	
	$R_{\max}$	R_{dist}	$R_{\max}$	R_{dist}
4	0.802	0.731	0.638	0.616
5	0.575	0.555	0.441	0.434
6	0.389	0.383	0.288	0.285
7	0.250	0.248	0.180	0.179
8	0.155	0.154	0.109	0.108
9	0.0930	0.0926	0.0641	0.0639
10	0.0547	0.0545	0.0371	0.0370

Similarly, $R_{\text{dist}} = (\log_q N_{\text{dist}})/2^m$ is the code rate of the distinct sequences in Construction 2, where N_{dist} is given in Lemma 6 and has been confirmed for small m's ($m = 4$ and 5 for binary case, and $m = 4$ for quaternary case) in numerical experiments. Table III shows that R_{dist} is close to $R_{\max}$ at every m, and it further approaches to $R_{\max}$ as m increases. It implies that the code rate loss induced by choosing distinct sequences is negligible for $m \geq 5$, and we are able to obtain a sufficiently large number of distinct sequences from Construction 2. Also, we observed that the code rate R_{dist} for a given m is higher than the ones in any other known near-complementary sequences [15][16][20].

Remark 1: When $t = 0$, we realize that the near-complementary sequences from Construction 2 are equivalent to the codewords in Golay complementary sets of size 4 and PMEPR ≤ 4 from Construction 14 in [20]. In the paper, the author indicated the potential of applying a permutation to the m variables in the Boolean function for increasing the number of codewords. However, he was concerned about a multiple generation of the same codewords, and the difficulty in removing the multiple codewords. Thus, he made no efforts for increasing the code rate (or number of sequences) by applying further permutations.

In Construction 2, we made a different approach $-$ a framework employing Golay complementary sequences as the seeds $-$ to construct the sequences for both $t = 0$ and 1. Moreover, we developed a generation algorithm of permutations for the sequences at $t = 1$. As a result, our sequence family contains the codewords from Construction 14 in [20] as its subset. The more sequences in the family enables the higher code rates than in [20].

V. NEAR COMPLEMENTARY SEQUENCES OF VARIOUS LENGTHS AND PMEPR < 4

This section applies the procedure described in Section III to provide new families of near-complementary sequences of various lengths and PMEPR < 4.

A. Shortened Golay complementary pair

Definition 2: Let $k \geq 2$ be a positive integer and $n = 2^k - 1$. Let g_1 and g_2 be the generalized Boolean functions given by

$$\begin{aligned} g_1(x_0, \cdots, x_{k-1}) &= \frac{q}{2} \sum_{r=0}^{k-2} x_{\pi_1(r)} x_{\pi_1(r+1)} + \sum_{r=0}^{k-1} u_r x_r + e, \quad u_r, e \in \mathbb{Z}_q, \\ g_2(x_0, \cdots, x_{k-1}) &= g_1(x_0, \cdots, x_{k-1}) + \frac{q}{2} x_{\pi_1(0)} \end{aligned} \tag{22}$$

where π_1 is a permutation in $\{0, \cdots, k-1\}$. While x runs through 0 to $2^k - 2$, the sequences $\mathbf{g}_1$ and $\mathbf{g}_2$ associated with g_1 and g_2 form a *shortened* Golay complementary pair of length $n = 2^k - 1$.

While x runs through all elements in $\mathbb{Z}_{2^k}$, the sequences $\widetilde{\mathbf{g}}_1$ and $\widetilde{\mathbf{g}}_2$ associated with (22) form a Golay complementary pair of length 2^k [16]. In fact, $\mathbf{g}_1$ and $\mathbf{g}_2$ are equivalently obtained by eliminating the last bits of the Golay complementary pair $\widetilde{\mathbf{g}}_1$ and $\widetilde{\mathbf{g}}_2$, respectively. In the following, we discuss the PMEPR of the shortened Golay pair.

Lemma 7: Let $G_1(z)$ and $G_2(z)$ be the associated polynomials of the shortened Golay complementary pair $\mathbf{g}_1$ and $\mathbf{g}_2$ of length $n = 2^k - 1$, respectively. Then, $\frac{|G_1(z)|^2 + |G_2(z)|^2}{n} < 4$.

Proof: Let $\widetilde{\mathbf{g}}_1 = (a_0, \cdots, a_n) = (\mathbf{g}_1, a_n)$ and $\widetilde{\mathbf{g}}_2 = (b_0, \cdots, b_n) = (\mathbf{g}_2, b_n)$ be the Golay complementary pair of length 2^k. Then, the aperiodic autocorrelations of $\mathbf{g}_1$ and $\mathbf{g}_2$ are given by

$$\rho_{\mathbf{g}_1}(\tau) = \rho_{\widetilde{\mathbf{g}}_1}(\tau) - \omega^{a_{n-\tau} - a_n}, \quad \rho_{\mathbf{g}_2}(\tau) = \rho_{\widetilde{\mathbf{g}}_2}(\tau) - \omega^{b_{n-\tau} - b_n}$$

where $\omega = e^{j\frac{2\pi}{q}}$. From (3),

$$\begin{aligned}
|G_1(z)|^2 + |G_2(z)|^2 &= 2n + \sum_{\tau=1}^{n-1} (\rho_{\mathbf{g}_1}(\tau) + \rho_{\mathbf{g}_2}(\tau)) z^{-\tau} + \sum_{\tau=1}^{n-1} (\rho^*_{\mathbf{g}_1}(\tau) + \rho^*_{\mathbf{g}_2}(\tau)) z^{\tau} \\
&= 2n + \sum_{\tau=1}^{n-1} (\rho_{\widetilde{\mathbf{g}}_1}(\tau) + \rho_{\widetilde{\mathbf{g}}_2}(\tau) - \omega^{a_{n-\tau}-a_n} - \omega^{b_{n-\tau}-b_n}) z^{-\tau} \\
&+ \sum_{\tau=1}^{n-1} (\rho^*_{\widetilde{\mathbf{g}}_1}(\tau) + \rho^*_{\widetilde{\mathbf{g}}_2}(\tau) - \omega^{-a_{n-\tau}+a_n} - \omega^{-b_{n-\tau}+b_n}) z^{\tau} \\
&= 2n - \left(\sum_{\tau=1}^{n-1} (\omega^{a_{n-\tau}-a_n} + \omega^{b_{n-\tau}-b_n}) z^{-\tau} + \sum_{\tau=1}^{n-1} (\omega^{-a_{n-\tau}+a_n} + \omega^{-b_{n-\tau}+b_n}) z^{\tau} \right) \\
&= 2n - 2 \sum_{\tau=1}^{n-1} \mathrm{Re}\big[(\omega^{a_{n-\tau}-a_n} + \omega^{b_{n-\tau}-b_n}) z^{-\tau} \big] \\
&\le 2n + 2 \sum_{\tau=1}^{n-1} \left| \omega^{a_\tau - a_n} + \omega^{b_\tau - b_n} \right|
\end{aligned}$$

where $\rho_{\widetilde{\mathbf{g}}_1}(\tau) + \rho_{\widetilde{\mathbf{g}}_2}(\tau) = \rho^*_{\widetilde{\mathbf{g}}_1}(\tau) + \rho^*_{\widetilde{\mathbf{g}}_2}(\tau) = 0$. From (22),

$$\begin{aligned}
b_\tau &= g_2(\tau_0, \cdots, \tau_{k-1}) = g_1(\tau_0, \cdots, \tau_{k-1}) + \frac{q}{2}\tau_{\pi_1(0)} = a_\tau + \frac{q}{2}\tau_{\pi_1(0)}, \\
b_n &= g_2(1, \cdots, 1) = g_1(1, \cdots, 1) + \frac{q}{2} = a_n + \frac{q}{2}
\end{aligned} \tag{23}$$

where $1 \le \tau = \sum_{l=0}^{k-1} \tau_l 2^l \le n-1, \ \tau_l \in \mathbb{Z}_2$. Finally,

$$\begin{aligned}
|G_1(z)|^2 + |G_2(z)|^2 &\le 2n + 2 \sum_{\tau=1}^{n-1} \left| \omega^{a_\tau - a_n} \right| \cdot \left| 1 - \omega^{\frac{q}{2}\tau_{\pi_1(0)}} \right| \\
&= 2n + 2 \cdot \mathcal{N}(\tau_{\pi_1(0)} = 1) \cdot 2 = 4n - 2 < 4n
\end{aligned}$$

where $\mathcal{N}(\tau_{\pi_1(0)} = 1) = \frac{n-1}{2}$ is the number of occurrences of $\tau_{\pi_1(0)} = 1$ while τ runs through 1 to $n-1$. $\square$

Applying the shortened Golay complementary pair as the seeds, the transformation procedure described in Section III leads us to the construction of a new family of near-complementary sequences of PMEPR < 4.

Construction 3: For $m \ge 3$, let $2 \le k \le m-1$ and $0 \le t \le m-k$. Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be the shortened Golay complementary pair of length $n = 2^k - 1$ in Definition 2. Employing the pair as the seeds and using Golay complementary sequences $\mathbf{c}$ of length $2^v = 2^{m-k}$, we construct a new family of near-complementary sequences $\mathbf{f}^{(t)}$ of length $n2^v = 2^m - 2^{m-k}$ by the procedure in Section III. Obviously, the PMEPR of $\mathbf{f}^{(t)}$ is less than 4 from Theorem 2 and Lemma 7.

As a matter of fact, the sequences in Example 1 are the near-complementary sequences of length 24 from Construction 3 ($m = 5$ and $k = 2$). Note that we may vary t, the seed pair $(\mathbf{g}_1, \mathbf{g}_2)$, and the Golay complementary sequence $\mathbf{c}$ in the procedure to generate a variety of sequences of length $2^m - 2^{m-k}$ and PMEPR < 4.

We can also use the generalized Boolean function in Theorem 2 for the equivalent construction of the near-complementary sequences.

Construction 4: Let m, k, n, and t be given in Construction 3. Let g_1 and g_2 be the generalized Boolean functions of k variables given in (22). Applying them to the general framework in Theorem 2, we have

$$\begin{aligned} f^{(t)}(x_0, \cdots, x_{m-1}) =& \frac{q}{2}\sum_{r=0}^{t-2} x_{\pi(r)}x_{\pi(r+1)} + \frac{q}{2}x_{\pi(t-1)}x_{\pi(t+k)} + \frac{q}{2}\sum_{r=t+k}^{m-2} x_{\pi(r)}x_{\pi(r+1)} \\ &+ \frac{q}{2}x_{\pi_1(t)}x_{\pi(\phi)} + \frac{q}{2}\sum_{r=t}^{t+k-2} x_{\pi_1(r)}x_{\pi_1(r+1)} + \sum_{r=0}^{m-1} u_r x_r + e \end{aligned} \tag{24}$$

where $u_r, e \in \mathbb{Z}_q$, and π_1 is a permutation in $\{t, t+1, \cdots, t+k-1\}$, while π is a permutation in $\{0, \cdots, t-1, t+k, \cdots, m-1\}$. In (24), $\phi = k$ if $t = 0$, and $\phi = 0$ otherwise. While $x = \sum_{l=0}^{m-1} x_l 2^l$ runs through the elements in $\mathbb{X}_t = \{x \mid x \in \mathbb{Z}_{2^m} \text{ where } 0 \le \sum_{l=0}^{k-1} x_{l+t}2^l \le 2^k - 2\}$ in the increasing order, the associated q-ary sequence $\mathbf{f}^{(t)}$ of length $2^m - 2^{m-k}$ is equivalent to the one in Construction 3.

In the generalized Boolean expression (24), if x runs through all elements in $\mathbb{Z}_{2^m}$, then $f^{(t)}$ generates Golay complementary sequences of length 2^m because the graph of the quadratic coset representatives forms a path [16]. It implies that $\mathbf{f}^{(t)}$ in Construction 4 can also be obtained by eliminating some parts of the Golay complementary sequences, where the parts correspond to the coordinates $x = \sum_{l=0}^{m-1} x_l 2^l$ with $(x_t, \cdots, x_{t+k-1}) = (1, \cdots, 1)$. It is easy to see that eliminating the parts is equivalent to removing the last column from the $2^{m-k} \times 2^k$ array in Table I of the $f^{(t)}$. Consequently, we consider another construction equivalent to Constructions 3 and 4.

Construction 5: Let $\check{\mathbf{f}}^{(t)} = (f_0, \cdots, f_{2^m-1})$ be the Golay complementary sequences of length 2^m associated with the generalized Boolean function of (24) while x runs through all elements in $\mathbb{Z}_{2^m}$. By eliminating the sequence elements of

$$f_j \text{ where } j = l \cdot 2^{k+t} + (2^k - 1)2^t + i, \quad 0 \le l \le 2^{m-k-t-1},\ 0 \le i \le 2^t - 1,$$

we construct the near-complementary sequences $\mathbf{f}^{(t)}$ of length $2^m - 2^{m-k}$ in Construction 4.

B. Extended Golay complementary pair

Definition 3: Let $h \ge 2$ be a positive integer and $n = 2^h + 1$. Let $\widetilde{\mathbf{g}}_1 = (a_0, \cdots, a_{n-2})$ and $\widetilde{\mathbf{g}}_2 = (b_0, \cdots, b_{n-2})$ be a Golay complementary pair of length 2^h associated with the generalized Boolean functions in (22) where $k = h$. For arbitrary $a_{n-1}, b_{n-1} \in \mathbb{Z}_q$, let $\mathbf{g}_1 = (\widetilde{\mathbf{g}}_1, a_{n-1})$ and $\mathbf{g}_2 = (\widetilde{\mathbf{g}}_2, b_{n-1})$. Then the pair $(\mathbf{g}_1, \mathbf{g}_2)$ is defined as an *extended* Golay complementary pair of length n.

Lemma 8: Let $G_1(z)$ and $G_2(z)$ be the associated polynomials of the extended Golay complementary pair $\mathbf{g}_1 = (\widetilde{\mathbf{g}}_1, a_{n-1})$ and $\mathbf{g}_2 = (\widetilde{\mathbf{g}}_2, b_{n-1})$ of length $n = 2^h + 1$, respectively, defined in Definition 3. If $b_{n-1} \in \{a_{n-1}, a_{n-1} + \frac{q}{2}\}$ for $a_{n-1} \in \mathbb{Z}_q$, then $\frac{|G_1(z)|^2 + |G_2(z)|^2}{n} < 4$.

Proof: From the extension, the aperiodic autocorrelations of $\mathbf{g}_1$ and $\mathbf{g}_2$ are given by

$$\rho_{\mathbf{g}_1}(\tau) = \rho_{\widetilde{\mathbf{g}}_1}(\tau) + \omega^{a_{n-\tau-1} - a_{n-1}}, \quad \rho_{\mathbf{g}_2}(\tau) = \rho_{\widetilde{\mathbf{g}}_2}(\tau) + \omega^{b_{n-\tau-1} - b_{n-1}}$$

where $\omega = e^{j\frac{2\pi}{q}}$. Thus,

$$\begin{aligned} |G_1(z)|^2 + |G_2(z)|^2 = 2n &+ \sum_{\tau=1}^{n-1} (\rho_{\widetilde{\mathbf{g}}_1}(\tau) + \rho_{\widetilde{\mathbf{g}}_2}(\tau) + \omega^{a_{n-\tau-1} - a_{n-1}} + \omega^{b_{n-\tau-1} - b_{n-1}}) z^{-\tau} \\ &+ \sum_{\tau=1}^{n-1} (\rho^*_{\widetilde{\mathbf{g}}_1}(\tau) + \rho^*_{\widetilde{\mathbf{g}}_2}(\tau) + \omega^{-a_{n-\tau-1} + a_{n-1}} + \omega^{-b_{n-\tau-1} + b_{n-1}}) z^{\tau} \\ &\le 2n + 2 \sum_{\tau=0}^{n-2} \left| \omega^{a_\tau - a_{n-1}} + \omega^{b_\tau - b_{n-1}} \right| \end{aligned}$$

where $\rho_{\widetilde{\mathbf{g}}_1}(\tau) + \rho_{\widetilde{\mathbf{g}}_2}(\tau) = \rho^*_{\widetilde{\mathbf{g}}_1}(\tau) + \rho^*_{\widetilde{\mathbf{g}}_2}(\tau) = 0$. If $b_{n-1} \in \{a_{n-1}, a_{n-1} + \frac{q}{2}\}$, then $\omega^{b_{n-1}} = (-1)^\delta \omega^{a_{n-1}}$ where $\delta = 0$ if $b_{n-1} = a_{n-1}$, and $\delta = 1$ otherwise. From (23),

$$\begin{aligned} |G_1(z)|^2 + |G_2(z)|^2 &\le 2n + 2 \sum_{\tau=0}^{n-2} \left| \omega^{a_\tau} + (-1)^\delta \omega^{b_\tau} \right| \\ &= 2n + 2 \sum_{\tau=0}^{n-2} \left| 1 + (-1)^\delta \omega^{\frac{q}{2}\tau_{\pi_1(0)}} \right| \\ &= 2n + 2 \cdot \mathcal{N}(\tau_{\pi_1(0)} = \delta) \cdot 2 = 4n - 2 < 4n \end{aligned}$$

where $\mathcal{N}(\tau_{\pi_1(0)} = \delta) = \frac{n-1}{2}$ is the number of occurrences of $\tau_{\pi_1(0)} = \delta \in \{0, 1\}$ for $0 \leq \tau \leq n-2$. $\square$

Construction 6: For $m \geq 3$, let $2 \leq h \leq m-1$ and $0 \leq t \leq m-h$. Let $\mathbf{g}_1 = (\widetilde{\mathbf{g}}_1, a_{n-1})$ and $\mathbf{g}_2 = (\widetilde{\mathbf{g}}_2, b_{n-1})$ be the extended Golay complementary pair of length $n = 2^h + 1$ in Definition 3, where $a_{n-1} \in \mathbb{Z}_q$ and $b_{n-1} \in \{a_{n-1}, a_{n-1} + \frac{q}{2}\}$. Employing the extended pair as the seeds and using Golay complementary sequences $\mathbf{c}$ of length $2^v = 2^{m-h}$, we construct a new family of near-complementary sequences $\mathbf{f}^{(t)}$ of length $n2^v = 2^m + 2^{m-h}$ by the procedure in Section III. Varying t from 0 to $m-h$, we can generate a variety of sequences $\mathbf{f}^{(t)}$ for different seed pairs $(\mathbf{g}_1, \mathbf{g}_2)$ and different $\mathbf{c}$'s in the procedure. Obviously, the PMEPR of $\mathbf{f}^{(t)}$ is less than 4.

In addition to Construction 6, we can consider the equivalent constructions by deriving the Boolean representation or appending 2^{m-h} elements to the Golay complementary sequences of length 2^m associated with (24). However, the constructions are straightforward and Construction 6 is enough to describe the near-complementary sequences employing the extended Golay pairs.

Remark 2: The Golay complementary pairs of length N do not exist if there is a prime p, $p \equiv 3 \mod 4$ such that $p \mid N$ [7]. Moreover, computer searches [4] refined the admissible sequence lengths for $N < 100$ by eliminating some lengths where no Golay pairs can be found. Finally, the sequence lengths of $N < 100$ where the Golay pairs exist are [12]

$$1, 2, 4, 8, 10, 16, 20, 26, 32, 40, 52, 64, 80.$$

In our near-complementary sequence family from shortened Golay pairs, the sequence length is $n2^{m-k}$ and $n = 2^k - 1 \equiv 3 \mod 4$ for $k \geq 2$. Therefore, if n is prime, the sequence family can provide an interesting construction example of sequences with low PMEPR for the lengths for which there is no explicit known construction of such sequences. For $N < 100$, our sequence families from shortened and extended Golay pairs also present many sequences with low PMEPR (< 4) for the lengths

$$24, 28, 30, 34, 36, 48, 56, 60, 62, 66, 68, 72, 96$$

where no Golay pairs exist.

TABLE IV

THE CODE RATES FOR NEAR-COMPLEMENTARY SEQUENCES FROM SHORTENED GOLAY PAIRS

m	k	length	Binary		Quaternary	
			R_s	$\mathrm{PMEPR}_{\max}$	R_s	$\mathrm{PMEPR}_{\max}$
5	2	24	0.483	3.327	0.366	3.332
	3	28	0.399	3.695	0.307	3.711
	4	30	0.386	3.333	0.293	3.333
6	2	48	0.311	3.333	0.228	
	3	56	0.253	3.711	0.189	
	4	60	0.236	3.310	0.176	
	5	62	0.240	2.845	0.177	
7	2	96	0.193	3.333	0.138	
	3	112	0.156	3.714	0.114	
	4	120	0.143	3.333	0.105	
	5	124	0.141	2.842	0.103	
	6	126	0.147	2.571	0.105	

C. Numerical results

We count the possible number of the distinct near-complementary sequences employing the shortened and the extended Golay pairs.

Let $N_k^{(s)}$ be the maximum number of the distinct near-complementary sequences of length $2^m - 2^{m-k}$ from Construction 3 (or equivalently Constructions 4, 5). Then,

$$\begin{aligned} N_k^{(s)} &= k! \cdot (m-k)! \cdot (m-k+1) \cdot q^{m+1} \\ &= k! \cdot (m-k+1)! \cdot q^{m+1}, \quad 2 \le k \le m-1 \end{aligned} \tag{25}$$

for all possible pairs ($\mathbf{g}_1$, $\mathbf{g}_2$), $\mathbf{c}$'s, and t's in the procedure. We expect that all the resulting sequences are distinct and total $N_k^{(s)}$ distinct sequences therefore exist for the length. However, it appears that the proof is not trivial. Instead, our experiments demonstrated that the number of the distinct sequences are in fact $N_k^{(s)}$ for all k's in $5 \le m \le 7$ for the binary case ($q = 2$), and for all k's in $m = 5$ for the quaternary case ($q = 4$). Table IV shows the code rates $R_s = \log_q N_k^{(s)}/(2^m - 2^{m-k})$ of our near-complementary sequences, where $N_k^{(s)}$ is given by (25) and $\mathrm{PMEPR}_{\max}$ is the maximum PMEPR measured over all the distinct sequences.

Let $N_h^{(e)}$ be the maximum number of the distinct near-complementary sequences of length

TABLE V

THE CODE RATES FOR NEAR-COMPLEMENTARY SEQUENCES FROM EXTENDED GOLAY PAIRS

m	h	length	Binary		Quaternary	
			R_e	$\mathrm{PMEPR}_{\max}$	R_e	$\mathrm{PMEPR}_{\max}$
5	2	40	0.340	3.600	0.257	3.600
	3	36	0.366	3.176	0.280	3.233
	4	34	0.400	2.941	0.303	2.941
6	2	80	0.211	3.594	0.156	
	3	72	0.225	3.223	0.168	
	4	68	0.238	2.917	0.178	
	5	66	0.256	2.685	0.189	
7	2	160	0.128	3.600	0.0922	
	3	144	0.135	3.223	0.0989	
	4	136	0.141	2.941	0.104	
	5	132	0.148	2.682	0.108	
	6	130	0.158	2.492	0.113	

$2^m + 2^{m-h}$ from extended Golay pairs. Similar to (25),

$$N_h^{(e)} = 2q \cdot h! \cdot (m-h+1)! \cdot q^{m+1}, \quad 2 \le h \le m-1. \tag{26}$$

The numerical experiments showed that the number of the distinct sequences are in fact $N_h^{(e)}$ for all h's in $m = 5, 6$, and for $h = 3, 4, 5$ in $m = 7$[1] in the binary case, and for all h's in $m = 5$ in the quaternary case. Table V shows the code rates $R_e = \log_q N_h^{(e)}/(2^m + 2^{m-h})$ of our near-complementary sequences, where $N_h^{(e)}$ is given by (26) and $\mathrm{PMEPR}_{\max}$ is the maximum PMEPR over all the distinct sequences.

In Tables IV and V, the code rates of the near-complementary sequences of length < 100 appear to be acceptable for the application to peak power control for a small number of subcarriers.

VI. CONCLUSION

This paper has presented a framework for near-complementary sequences of length $n2^{m-k}$ by the generalized Boolean representation and the equivalent array structure. The framework

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[1] For $h = 2$ and 6 in $m = 7$, the experiments are progressing.

transforms the seed pairs of length n to near-complementary sequences of length $n2^{m-k}$ by the aid of Golay complementary sequences of length 2^{m-k}, preserving the PMEPR bound of the seeds. Employing Golay complementary sequences of length 2^k as the seeds, we have constructed a family of near-complementary sequences of length 2^m and PMEPR ≤ 4. With $k = m - 1$ and novel permutations, the family has provided more distinct sequences than any other known near-complementary sequences.

Furthermore, we have constructed new families of near-complementary sequences of various lengths by applying the shortened and the extended Golay complementary pairs to the framework as the seeds. A large number of sequences with low PMEPR < 4 could be found for the particular lengths (< 100), i.e., $24, 28, 30, 34, 36, 48, 56, 60, 62, 66, 68, 72$, and 96 for which there have been no explicit constructions of Golay complementary sequences. In practice, the sequence families can give the flexibility to coding solutions for peak power control in multicarrier communications by providing many codewords of various lengths and low PMEPR.

In [13], the authors pointed out that the full usefulness of the seed technique will be apparent only if a method can be found to construct seeds (as opposed to computational search). They left the method as an open problem, which has been partly solved by the construction example of near-complementary sequences in [19]. In Construction 2, we could also solve the problem by providing a large number of distinct near-complementary sequences of length 2^m using the seed technique in a constructive way. In Constructions 3 and 6, we further exploited the usefulness of the seed technique by employing the shortened and the extended Golay pairs and producing many near-complementary sequences of various lengths.

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